Connection theory on differentiable fibre bundles:

A pedagogical introduction

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Abstract

The paper contains a partial review on the general connection theory on differentiable fibre bundles. Particular attention is paid on (linear) connections on vector bundles. The (local) representations of connections in frames adapted to holonomic and arbitrary frames is considered.

1. Introduction

This is a partial review of the connection theory on differentiable fibre bundles. From different view-points, this theory can be found in many works, like [1–21]. The presentation of the material in sections 2–5, containing the grounds of the connection theory, follows some of the main ideas of [20, chapters 1 and 2], but their realization here is quite different and follows the modern trends in differential geometry. Since in the physical literature one can find misunderstanding or not quite rigorous applications of known mathematical definitions and results, the text is written in a way suitable for direct application in some regions of theoretical physics.

The work is organized as follows.

In Sect. 2 is collected some introductory material, like the notion of Lie derivatives and distributions on manifolds, needed for our exposition. Here some of our notation is fixed too.

Section 3 is devoted to the general connection theory on bundles whose base and bundles spaces are differentiable manifolds. In Subsect. 3.1 are reviewed some coordinates and frames/bases on the bundle space which are compatible with the fibre structure of a bundle. Subsect. 3.2 deals with the general connection theory. A connection on a bundle is defined as a distribution on its bundle space which is complimentary to the vertical distribution on it. The notion of parallel transport generated by connection and of specialized frame are introduced. The fibre coefficients and fibre components of the curvature of a connection are defined via part of the components of the anholonomy object of a specialized frame. Frames adapted to local bundle coordinates are introduced and the local (2-index) coefficients in them of a connection are defined; their transformation law is derived and it is proved that a geometrical object with such transformation law uniquely defines a connection. The parallel transport equation in their terms is derived and it is demonstrated how from it the equation of geodesics on a manifold can be obtained.

In Sect. 4, the general connection theory from Sect. 3 is specified on vector bundles. The most important structures in/on them are the ones that are consistent/compatible with the vector space structure of their fibres. The vertical lifts of sections of a vector bundle and the horizontal lifts of vector fields on its base are investigated in more details in Subsect. 4.1. The general results are specified on the (co)tangent bundle over a manifold in Subsect. 4.2. Subsect. 4.3 is devoted to linear connections on vector bundles, i.e. connections such that the assigned to them parallel transport is a linear mapping. It is proved that the 2-index coefficients of a linear connection are linear in the fibre coordinates, which leads to the introduction of the (3-index) coefficients of the connection; the latter coefficients being defined on the base space. The transformations of different objects under a change of vector bundle coordinates are explored. The covariant derivatives are introduced and investigated in Subsect. 4.4. They are defined via the Lie derivatives and a mapping realizing an isomorphism between the vertical vector fields on the bundle space and the sections of the bundle. The equivalence of that definition with the widespread one, defining them as mappings on the module of sections of the bundle with suitable properties, is proved. Some properties of the covariant derivatives are explored. In Subsect. 4.5, the affine connections on vector bundles are considered briefly.

Section 5 deals briefly with morphisms between bundles with connections defined on them.

In section 6, some of the results of the previous sections are generalized when frames more general than the ones generated by local coordinates on the bundle space are employed. The most general such frames, compatible with the fibre structure, and the frames adapted to them are investigated. The main differential-geometric objects, introduced in the previous sections, are considered in such general frames. Particular attention is paid to the case of a vector bundle. In vector bundles, a bijective correspondence between the mentioned

general frames and pairs of bases, in the vector fields over the base and in the sections of the bundle, is proved. The (3-index) coefficients of a connection in such pairs of frames and their transformation laws are considered. The covariant derivatives are also mentioned on this context.

Section 7 closes the paper with some concluding remarks.

2. Preliminaries

This section contains an introductory material, notation etc. that will be needed for our exposition. The reader is referred for details to standard books on differential geometry, like [5, 22, 23].

A differentiable finite-dimensional manifold over a field $\mathbb{K}$ will be denoted typically by M. Here $\mathbb{K}$ stands for the field $\mathbb{R}$ of real or the field $\mathbb{C}$ of complex numbers, $\mathbb{K} = \mathbb{R}, \mathbb{C}$. The manifolds we consider are supposed to be smooth of class C^2. [1] The sets of vector fields, realized as first order differential operators, and of differential k-forms, $k \in \mathbb{N}$, over M will be denoted by $\mathcal{X}(M)$ and $\Lambda^k(M)$, respectively. The space tangent (resp. cotangent) to M at $p \in M$ is $T_p(M)$ (resp. $T_p^*(M)$) and $(T(M), \pi_T, M)$ (resp. $(T^*(M), \pi_{T^*}, M)$) will stand for the tangent (resp. cotangent) bundle over M. The value of $X \in \mathcal{X}(M)$ at $p \in M$ is $X_p \in T_p(M)$ and the action of X on a C^1 function $\varphi\colon M \to \mathbb{K}$ is a function $X(\varphi)\colon M \to \mathbb{K}$ with $X(\varphi)|_p := X_p(\varphi) \in \mathbb{K}$.

If M and $\bar{M}$ are manifolds and $f\colon \bar{M} \to M$ is a C^1 mapping, then $f_* := \mathrm{d}f := T(f)\colon T(\bar{M}) \to T(M)$ denotes the induced tangent mapping (or differential) of f such that, for $p \in M$, $f_*|_p := \mathrm{d}f|_p := T_p(f)\colon T_p(\bar{M}) \to T_{f(p)}(M)$ and, for a C^1 function g on M, $(f_*(X))(g) := X(g \circ f)\colon p \mapsto f_*|_p(g) = X_p(g \circ f)$, with $\circ$ being the composition of mappings sign. Respectively, the induced cotangent mapping is $f^* := T^*(f)\colon T^*(M) \to T^*(\bar{M})$. If $h\colon N \to \bar{M}$, N being a manifold, we have the chain rule $\mathrm{d}(f \circ h) = \mathrm{d}f \circ \mathrm{d}h$, which is an abbreviation for $\mathrm{d}(f \circ h)_q = (\mathrm{d}f)_{f(q)} \circ (\mathrm{d}h)_q$ for $q \in N$.

By $J \subseteq \mathbb{R}$ will be denoted an arbitrary real interval that can be open or closed at one or both its ends. The notation $\gamma\colon J \to M$ represents an arbitrary path in M. For a C^1 path $\gamma\colon J \to M$, the vector tangent to γ at $s \in J$ will be denoted by $\dot\gamma(s) := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=s}(\gamma(t)) = \gamma_*\big(\frac{\mathrm{d}}{\mathrm{d}r}\big|_s\big) \in T_{\gamma(s)}(M)$, where r in $\frac{\mathrm{d}}{\mathrm{d}r}\big|_s$ is the standard coordinate function on $\mathbb{R}$, i.e. $r\colon \mathbb{R} \to \mathbb{R}$ with $r(s) := s$ for all $s \in \mathbb{R}$ and hence $r = \mathsf{id}_{\mathbb{R}}$ is the identity mapping of $\mathbb{R}$. If $s_0 \in J$ is an end point of J and J is closed at s_0, the derivative in the definition of $\dot\gamma(s_0)$ is regarded as a one-sided derivative at s_0.

The Lie derivative relative to $X \in \mathcal{X}(M)$ will be denoted by $\mathcal{L}_X$. It is defined on arbitrary geometrical objects on M [24], but bellow we shall be interested in its action on tensor fields [2, ch. I, § 2] (see also [25]). If f, Y, and θ are C^1 respectively function, vector field and 1-form on M, then

$$\mathcal{L}_X(f) = X(f) \tag{2.1a}$$
$$\mathcal{L}_X(Y) = [X, Y]_{\_} \tag{2.1b}$$
$$(\mathcal{L}_X(\theta))(Y) = X(\theta(Y)) - \theta([X,Y]_{\_}) = (\mathrm{d}\theta)(X,Y) + Y(\theta(X)), \tag{2.1c}$$

where $[A, B]_{\_} = A \circ B - B \circ A$ is the commutator of operators A and B (with common domain) and d denotes the exterior derivative operator.

[1] Some of our definitions or/and results are valid also for C^1 or even C^0 manifolds, but we do not want to overload the material with continuous counting of the required degree of differentiability of the manifolds involved. Some parts of the text admit generalizations on more general spaces, like the topological ones, but this is out of the subject of the present work.

Since $\mathcal{L}_X$ is a derivation of the tensor algebra over the vector fields on M, for a tensor field $S\colon \Lambda^1(M)\times\dots\times\Lambda^1(M)\times\mathcal{X}(M)\times\dots\times\mathcal{X}(M)$, we have

$$(\mathcal{L}_X S)(\theta,\dots;Y,\dots)=X(S(\theta,\dots;Y,\dots))-S(\mathcal{L}_X\theta,\dots;Y,\dots)-\dots-S(\theta,\dots;\mathcal{L}_X Y,\dots))-\cdots, \tag{2.2}$$

which defines $\mathcal{L}_X S$ explicitly, due to (2.1).

Let the Greek indices $\lambda,\mu,\nu,\dots$ run over the range $1,\dots,\dim M$ and $\{E_\mu\}$ be a C^1 frame in $T(M)$, i.e. $E_\mu\in\mathcal{X}(M)$ be of class C^1 and, for each $p\in M$, the set $\{E_\mu|_p\}$ be a basis of the vector space $T_p(M)$. [2] Let $\{E^\mu\}$ be the coframe dual to $\{E_\mu\}$, i.e. $E^\mu\in\Lambda^1(M)$, $\{E^\mu|_p\}$ be a basis in $T_p^*(M)$, and $E^\mu(E_\nu)=\delta^\mu_\nu$ with δ^ν_μ being the Kronecker deltas ($\delta^\nu_\mu=1$ for $\mu=\nu$ and $\delta^\nu_\mu=0$ for $\mu\neq\nu$). Assuming the Einstein's summation convention (summation on indices repeated on different levels over the whole range of their values), we define the *components* $(\Gamma_X)^\mu{}_\nu$ of $\mathcal{L}_X$ in (relative to) $\{E_\mu\}$ via the expansion

$$\mathcal{L}_X E_\mu =: (\Gamma_X)^\nu{}_\mu E_\nu \tag{2.3}$$

which is equivalent to

$$\mathcal{L}_X E^\mu = -(\Gamma_X)^\mu{}_\nu E^\nu \tag{2.3$'$}$$

by virtue of $E^\mu(E_\nu)=\delta^\mu_\nu$ and the commutativity of the Lie derivatives and contraction operators. [3] Sometimes, it is convenient (2.3) and (2.3$'$) to be written in a matrix form,

$$\mathcal{L}_X E = E\cdot\Gamma_X \qquad \mathcal{L}_X E^* = -\Gamma_X\cdot E^*, \tag{2.4}$$

where $\Gamma_X := [(\Gamma_X)^\mu{}_\nu]_{\mu,\nu=1}^{\dim M}$, $E:=(E_1,\dots,E_{\dim M})$, and $E^*:=(E^!,\dots,E^{\dim M})^\top$, with $\top$ being the matrix transposition sign, and the matrix multiplication is explicitly denoted by centered dot $\cdot$ as otherwise $E\cdot\Gamma_X$ may be confused with $E\Gamma_X=E(\Gamma_X)=(E_1(\Gamma_X),\dots)=([E_1((\Gamma_X)^\nu{}_\mu)],\dots)$. From (2.3) and (2.1b), we get

$$(\Gamma_X)^\nu{}_\mu = -E_\mu(X^\nu) - C^\nu_{\mu\lambda}X^\lambda, \tag{2.5}$$

in $\{E_\mu\}$, where $X=X^\mu E_\mu$ and the functions $C^\nu_{\mu\lambda}$, known as the *components of the anholonomy object of* $\{E_\mu\}$, are defined by

$$[E_\mu,E_\nu]_\_ =: C^\lambda_{\mu\nu}E_\lambda \tag{2.6}$$

or, equivalently, by its dual (see (2.1c))

$$dE^\lambda = -\frac{1}{2}C^\lambda_{\mu\nu}E^\mu\wedge E^\nu, \tag{2.6$'$}$$

with $\wedge$ being the exterior (wedge) product sign. [4] For a tensor field S of type (r,s), $r,s\in\mathbb{N}\cup\{0\}$, with components $S^{\mu_1,\dots,\mu_r}_{\nu_1,\dots,\nu_s}$ relative to the tensor frame induced by $\{E_\mu\}$ and $\{E^\mu\}$,

[2] There are manifolds, like the even-dimensional spheres $\mathbb{S}^{2k}$, $k\in\mathbb{N}$, which do not admit global, continuous (and moreover C^k for $k\geq1$), and nowhere vanishing vector fields [26]. If this is the case, the considerations must be localized over an open subset of M on which such fields exist. We shall not overload our exposition with such details.

[3] The sign before $(\Gamma_X)^\mu{}_\nu$ in (2.3) or (2.3$'$) is conventional and we have chosen it in a way similar to the accepted convention for the components of a covariant derivative (or, equivalently, the coefficients of a linear connection — see Sect. 4).

[4] If M is a Lie group and $\{E_\mu\}$ is a basis of its Lie algebra (:= {left invariant vector fields in $\mathcal{X}(M)$}), then $C^\lambda_{\mu\nu}$ are constants, called structure constants of M, and (2.6) and (2.6$'$) are known as the structure equations of M.

we get, from (2.2), the components of $\mathcal{L}_X S$ as

$$\big(\mathcal{L}_X S\big)^{\mu_1,\dots,\mu_r}_{\nu_1,\dots,\nu_s} = X\big(S^{\mu_1,\dots,\mu_r}_{\nu_1,\dots,\nu_s}\big) + \sum_{a=1}^{r} (\Gamma_X)^{\mu_a}{}_{\lambda} S^{\mu_1,\dots,\mu_{a-1},\lambda,\mu_{a+1},\dots,\mu_r}_{\nu_1,\dots,\nu_s} \\ - \sum_{b=1}^{s} (\Gamma_X)^{\lambda}{}_{\nu_b} S^{\mu_1,\dots,\mu_r}_{\nu_1,\dots,\nu_{b-1},\lambda,\nu_{b+1},\dots,\nu_s}. \quad (2.7)$$

A frame $\{E_\mu\}$ or its dual coframe $\{E^\mu\}$ is called *holonomic* (*anholonomic*) if $C^\lambda_{\mu\nu} = 0$ ($C^\lambda_{\mu\nu} \neq 0$) for all (some) values of the indices μ, ν, and λ. For a holonomic frame always exist local coordinates $\{x^\mu\}$ on M such that *locally* $E_\mu = \frac{\partial}{\partial x^\mu}$ and $E^\mu = \mathrm{d}x^\mu$. Conversely, if $\{x^\mu\}$ are local coordinates on M, then the local frame $\{\frac{\partial}{\partial x^\mu}\}$ and local coframe $\{\mathrm{d}x^\mu\}$ are defined and holonomic on the domain of $\{x^\mu\}$.

A straightforward calculation by means of (2.6) reveals that a change

$$\{E_\mu\} \to \{\bar{E}_\mu = B^\nu_\mu E_\nu\} \quad (2.8)$$

of the frame $\{E_\mu\}$, where $B = [B^\nu_\mu]$ is a non-degenerate matrix-valued function, entails the transformation

$$C^\lambda_{\mu\nu} \mapsto \bar{C}^\lambda_{\mu\nu} = (B^{-1})^\lambda_\varrho \big(B^\sigma_\mu E_\sigma(B^\varrho_\nu) - B^\sigma_\nu E_\sigma(E^\varrho_\mu) + B^\sigma_\mu B^\tau_\nu C^\varrho_{\sigma\tau}\big). \quad (2.9)$$

Besides, from (2.5) and (2.9), we see that the quantities $(\Gamma_X)^\nu{}_\mu$ undergo the change

$$(\Gamma_X)^\nu{}_\mu \mapsto (\bar{\Gamma}_X)^\nu{}_\mu = (B^{-1})^\mu_\varrho \big((\Gamma_X)^\varrho{}_\sigma B^\sigma_\nu + X(B^\sigma_\nu)\big) \quad (2.10)$$

when (2.8) takes place. Setting $\Gamma_X := [(\Gamma_X)^\nu{}_\mu]$ and $\bar{\Gamma}_X := [(\bar{\Gamma}_X)^\nu{}_\mu]$, we can rewrite (2.10) in a more compact matrix form as

$$\Gamma_X \mapsto \bar{\Gamma}_X = B^{-1} \cdot (\Gamma_X \cdot B + X(B)). \quad (2.11)$$

If $n \in \mathbb{N}$ and $n \leq \dim M$, an n-dimensional *distribution* Δ on M is defined as a mapping $\Delta\colon p \mapsto \Delta_p$ assigning to each $p \in M$ an n-dimensional subspace Δ_p of the tangent space $T_p(M)$ of M at p, $\Delta_p \subseteq T_p(M)$. A *solution* (resp. *first integral*) of a distribution Δ on M is an immersion $\varphi\colon N \to M$ (resp. submersion $\psi\colon M \to N$), N being a manifold, such that $\operatorname{Im}\varphi_* \subseteq \Delta$ (resp. $\operatorname{Ker}\psi_* \supset \Delta$), i.e., for each $q \in N$, (resp. $p \in M$), $\varphi_*(T_q(N)) \subseteq \Delta_{\varphi(q)}$ (resp. $\psi_*(\Delta_p) = 0_{\psi(q)} \in T_{\psi(q)}(N)$). A distribution is *integrable* if there is a submersion $\psi\colon M \to N$ such that $\operatorname{Ker}\psi_* = \Delta$; a necessary and locally sufficient condition for the integrability of Δ is the commutator of every two vector fields in Δ to be in Δ. We say that a vector field $X \in \mathcal{X}(M)$ is in Δ and write $X \in \Delta$, if $X_p \in \Delta_p$ for all $p \in M$. A *basis on* $U \subseteq M$ *for* Δ is a set $\{X_1,\dots,X_n\}$ of n linearly independent (relative to functions $U \to \mathbb{K}$) vector fields in $\Delta|_U$, i.e. $\{X_1|_p,\dots,X_n|_p\}$ is a basis for Δ_p for all $p \in U$.

A distribution is convenient to be described in terms of (global) frames or/and coframes over M. In fact, if $p \in M$ and $\varrho = 1,\dots,n$, in each $\Delta_p \subseteq T_p(M)$, we can choose a basis $\{X_\varrho|_p\}$ and hence a frame $\{X_\varrho\}$, $X_\varrho\colon p \mapsto X_\varrho|_p$, in $\{\Delta_p : p \in M\} \subseteq T(M)$; we say that $\{X_\rho\}$ is a basis for/in Δ. Conversely, any collection of n linearly independent (relative to functions $M \to \mathbb{K}$) vector fields X_ϱ on M defines a distribution $p \mapsto \big\{\sum_{\varrho=1}^n f^\varrho X_\varrho|_p : f^\varrho \in \mathbb{K}\big\}$. Consequently, a frame in $T(M)$ can be formed by adding to a basis for Δ a set of $(\dim M - n)$ new linearly independent vector fields (forming a frame in $T(M) \setminus \{\Delta_p : p \in M\}$) and v.v., by selecting n linearly independent vector fields on M, we can define a distribution Δ on M. Equivalently, one can use $\dim M - n$ linearly independent 1-forms ω^a, $a = n+1,\dots,\dim M$, which are annihilators for it, $\omega^a|_{\Delta_p} = 0$ for all $p \in M$. For instance, if $\{X_\mu : \mu = 1,\dots,\dim M\}$ is a frame in $T(M)$ and $\{X_\varrho : \varrho = 1,\dots,n\}$ is a basis for Δ, then one can define ω^a to be elements in the coframe $\{\omega^\mu\}$ dual to $\{X_\mu\}$. We call $\{\omega^a\}$ a *cobasis* for Δ.

3. Connections on bundles

Before presenting the general connection theory in Subsect. 3.2, we at first fix some notation and concepts concerning fibre bundles in Subsect. 3.1.

3.1. Frames and coframes on the bundle space

Let (E, π, M) be a bundle with bundle space E, projection $\pi\colon E \to M$, and base space M. Suppose that the spaces M and E are manifolds of finite dimensions $n \in \mathbb{N}$ and $n + r$, for some $r \in \mathbb{N}$, respectively; so the dimension of the fibre $\pi^{-1}(x)$, with $x \in M$, i.e. the fibre dimension of (E, π, M), is r. Besides, let these manifolds be C^1 differentiable, if the opposite is not stated explicitly. [5]

Let the Greek indices $\lambda, \mu, \nu, \dots$ run from 1 to $n = \dim M$, the Latin indices $a, b, c, \dots$ take the values from $n+1$ to $n+r = \dim E$, and the uppercase Latin indices $I, J, K, \dots$ take values in the whole set $\{1, \dots, n+r\}$. One may call these types of indices respectively base, fibre, and bundle indices.

Suppose $\{u^I\} = \{u^\mu, u^a\} = \{u^1, \dots, u^{n+r}\}$ are local *bundle coordinates* on an open set $U \subseteq E$, i.e. on the set $\pi(U) \subseteq M$ there are local coordinates $\{x^\mu\}$ such that $u^\mu = x^\mu \circ \pi$; [6] the coordinates $\{u^\mu\}$ (resp. $\{u^a\}$) are called *basic* (resp. *fibre*) *coordinates* [23]. [7]

Further only coordinate changes

$$\{u^\mu, u^a\} \mapsto \{\tilde{u}^\mu, \tilde{u}^a\} \tag{3.1a}$$

on E which respect the fibre structure, viz. the division into basic and fibre coordinates, will be considered. This means that

$$\begin{aligned} \tilde{u}^\mu(p) &= f^\mu(u^1(p), \dots, u^n(p)) \\ \tilde{u}^a(p) &= f^a(u^1(p), \dots, u^n(p), u^{n+1}(p), \dots, u^{n+r}(p)) \end{aligned} \tag{3.1b}$$

for $p \in E$ and some functions f^I. The bundle coordinates $\{u^\mu, u^a\}$ induce the (local) frame $\{\partial_\mu := \frac{\partial}{\partial u^\mu}, \partial_a := \frac{\partial}{\partial u^a}\}$ and coframe $\{\mathrm{d}u^\mu, \mathrm{d}u^a\}$ over U in respectively the tangent $T(E)$ and cotangent $T^*(E)$ bundle spaces of the tangent and cotangent bundles over the bundle space E. Since a change (3.1) of the coordinates on E implies $\partial_I \mapsto \tilde{\partial}_I := \frac{\partial}{\partial \tilde{u}^I} = \frac{\partial u^J}{\partial \tilde{u}^I} \partial_J$ and $\mathrm{d}u^I \mapsto \partial \tilde{u}^I = \frac{\partial \tilde{u}^I}{\partial u^J} \mathrm{d}u^J$, the transformation (3.1) leads to

$$(\partial_\mu, \partial_a) \mapsto (\tilde{\partial}_\mu, \tilde{\partial}_a) = (\partial_\nu, \partial_b) \cdot A \tag{3.2a}$$

$$(\mathrm{d}u^\mu, \mathrm{d}u^a)^\top \mapsto (\mathrm{d}\tilde{u}^\mu, \mathrm{d}\tilde{u}^a)^\top = A^{-1} \cdot (\mathrm{d}u^\nu, \mathrm{d}u^b)^\top. \tag{3.2b}$$

Here expressions like $(\partial_\mu, \partial_a)$ are shortcuts for ordered $(n+r)$-tuples like $(\partial_1, \dots, \partial_{n+r}) = \big([\partial_\mu]_{\mu=1}^n, [\partial_a]_{a=n+1}^{n+r}\big)$, $\top$ is the matrix transpositions sign, the centered dot $\cdot$ stands for the matrix multiplication, and the transformation matrix A is

$$A := \left[\frac{\partial u^I}{\partial \tilde{u}^J}\right]_{I,J=1}^{n+r} = \begin{pmatrix} \left[\frac{\partial u^\nu}{\partial \tilde{u}^\mu}\right] & 0_{n\times r} \\ \left[\frac{\partial u^b}{\partial \tilde{u}^\mu}\right] & \left[\frac{\partial u^b}{\partial \tilde{u}^a}\right] \end{pmatrix} =: \begin{bmatrix} \frac{\partial u^\nu}{\partial \tilde{u}^\mu} & 0 \\ \frac{\partial u^b}{\partial \tilde{u}^\mu} & \frac{\partial u^b}{\partial \tilde{u}^a} \end{bmatrix}, \tag{3.3}$$

[5] Most of our considerations are valid also if C^1 differentiability is assumed and even some of them hold on C^0 manifolds. By assuming C^2 differentiability, we skip the problem of counting the required differentiability class of the whole material that follows. Sometimes, the C^2 differentiability is required explicitly, which is a hint that a statement or definition is not valid otherwise. If we want to emphasize that some text is valid under a C^1 differentiability assumption, we indicate that fact explicitly.

[6] On a bundle or fibred manifold, these coordinates are known also as adapted coordinates [27, definition 1.1.5].

[7] If (U, v) is a bundle chart, with $v\colon U \to \mathbb{K}^n \times \mathbb{K}^r$ and $e^a\colon \mathbb{K}^r \to \mathbb{K}$ are such that $e^a(c_1, \dots, c_r) = c_a \in \mathbb{K}$, then one can put $u^a = e^a \circ \mathrm{pr}_2 \circ v$, where $\mathrm{pr}_2\colon \mathbb{K}^n \times \mathbb{K}^r \to \mathbb{K}^r$ is the projection on the second multiplier $\mathbb{K}^r$.

where $0_{n\times r}$ is the $n\times r$ zero matrix. Besides, in expressions of the form $\partial_I a^I$, like the one in the r.h.s. of (3.2a), the summation excludes differentiation, i.e. $\partial_I a^I := a^I\partial_I = \sum_I a^I\partial_I$; if a differentiation really takes place, we write $\partial_I(a^I) := \sum_I \partial_I(a^I)$. This rule allows a lot of formulae to be writtem in a compact matrix form, like (3.2a). The explicit form of the matrix inverse to (3.3) is $A^{-1} = \left[\frac{\partial\tilde{u}^I}{\partial u^J}\right] = \dots$ and it is obtained from (3.3) via the change $u \leftrightarrow \tilde{u}$.

The formulae (3.2) can be generalized for arbitrary frame $\{e_I\} = \{e_\mu, e_a\}$ in $T(E)$ and its dual coframe $\{e^I\} = \{e^\mu, e^a\}$ in $T^*(E)$ which respect the fibre structure in a sense that their *admissible changes* are given by

$$(e_I) = (e_\mu, e_a) \mapsto (\tilde{e}_I) = (\tilde{e}_\mu, \tilde{e}_a) = (e_\nu, e_b)\cdot A \tag{3.4a}$$

$$\begin{pmatrix} e^\mu \\ e^a \end{pmatrix} \mapsto \begin{pmatrix} \tilde{e}^\mu \\ \tilde{e}^a \end{pmatrix} = A^{-1}\cdot\begin{pmatrix} e^\nu \\ e^b \end{pmatrix}. \tag{3.4b}$$

Here $A = [A^I_J]$ is a nondegenerate matrix-valued function with a block structure similar to (3.3), viz.

$$A = \begin{pmatrix} [A^\nu_\mu]^n_{\mu,\nu=1} & 0_{n\times r} \\ [A^b_\mu]_{\substack{\mu=1,\dots,n \\ b=n+1,\dots,n+r}} & [A^b_a]^{n+r}_{a,b=n+1} \end{pmatrix} =: \begin{bmatrix} A^\nu_\mu & 0 \\ A^b_\mu & A^b_a \end{bmatrix} \tag{3.5a}$$

with inverse matrix

$$A^{-1} = \begin{pmatrix} [A^\nu_\mu]^{-1} & 0 \\ -[A^a_b]^{-1}\cdot[A^a_\mu]\cdot[A^\nu_\mu]^{-1} & [A^a_b]^{-1} \end{pmatrix}. \tag{3.5b}$$

Here $A^a_\mu\colon U \to \mathbb{K}$ and $[A^\nu_\mu]$ and $[A^a_b]$ are non-degenerate matrix-valued functions on U such that $[A^\nu_\mu]$ is constant on the fibres of E, i.e., for $p\in E$, $A^\nu_\mu(p)$ depends only on $\pi(p)\in M$, which is equivalent to any one of the equations

$$A^\nu_\mu = B^\nu_\mu\circ\pi \qquad \frac{\partial A^\nu_\mu}{\partial u^a} = 0, \tag{3.6}$$

with $[B^\nu_\mu]$ being a nondegenerate matrix-valued function on $\pi(U)\subseteq M$. Obviously, (3.2) corresponds to (3.4) with $e_I = \frac{\partial}{\partial u^I}$, $\tilde{e}_I = \frac{\partial}{\partial\tilde{u}^I}$, and $A^J_I = \frac{\partial u^J}{\partial\tilde{u}^I}$.

All frames on E connected via (3.4)–(3.5), which are (locally) obtainable from holonomic ones, induced by bundle coordinates, via admissible changes, will be referred as *bundle frames*. Only such frames will be employed in the present work.

If we deal with a *vector* bundle (E,π,M) endowed with vector bundle coordinates $\{u^I\}$ [23], then the new fibre coordinates $\{\tilde{u}^a\}$ in (3.1) must be *linear and homogeneous* in the old ones $\{u^a\}$, i.e.

$$\tilde{u}^a = (B^a_b\circ\pi)\cdot u^b \text{ and } u^a = ((B^{-1})^a_b\circ\pi)\cdot\tilde{u}^b, \tag{3.7}$$

with $B = [B^a_b]$ being a non-degenerate matrix-valued function on $\pi(U)\subseteq M$. In that case, the matrix (3.3) and its inverse take the form

$$A = \begin{bmatrix} \frac{\partial u^\mu}{\partial\tilde{u}^\nu} & 0 \\ \left(\frac{\partial(B^{-1})^a_b}{\partial\tilde{x}^\nu}\circ\pi\right)\cdot\tilde{u}^b & (B^{-1})^b_a\circ\pi \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} \frac{\partial\tilde{u}^\nu}{\partial u^\mu} & 0 \\ \left(\frac{\partial B^b_a}{\partial x^\mu}\circ\pi\right)\cdot u^a & B^b_a\circ\pi \end{bmatrix}. \tag{3.8}$$

More generally, in the vector bundle case, admissible are transformations (3.4) with matrices like

$$A = \begin{pmatrix} [A^\mu_\nu] & 0 \\ [A^b_{c\mu}\tilde{u}^c] & [A^a_b] \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} [A^\mu_\nu]^{-1} & 0 \\ -[A^a_b]^{-1}\cdot[A^b_{c\mu}\tilde{u}^c]\cdot[A^\mu_\nu]^{-1} & [A^a_b]^{-1} \end{pmatrix} \tag{3.9}$$

with $A^a_{b\mu}\colon U \to \mathbb{K}$ being functions on U which are constant on the fibres of E,

$$A^a_{b\mu} = B^a_{b\mu} \circ \pi \qquad \frac{\partial A^a_{b\mu}}{\partial u^c} = 0 \tag{3.10}$$

for some functions $B^a_{b\mu}\colon \pi(U) \to \mathbb{K}$. Obviously, (3.9) corresponds to (3.5) with $A^b_\mu = A^b_{c\mu}\tilde{u}^c$ and the setting $A^J_I = \frac{\partial u^J}{\partial \tilde{u}^I}$. reduces (3.9) to (3.8) due to (3.7).

3.2. Connection theory

From a number of equivalent definitions of a connection on differentiable manifold [21, sections 2.1 and 2.2], we shall use the following one.

Definition 3.1. A *connection on a bundle* (E, π, M) is an $n = \dim M$ dimensional distribution Δ^h on E such that, for each $p \in E$ and the *vertical distribution* Δ^v defined by

$$\Delta^v\colon p \mapsto \Delta^v_p := T_{\imath(p)}\big(\pi^{-1}(\pi(p))\big) \cong T_p\big(\pi^{-1}(\pi(p))\big), \tag{3.11}$$

with $\imath\colon \pi^{-1}(\pi(p)) \to E$ being the inclusion mapping, is fulfilled

$$\Delta^v_p \oplus \Delta^h_p = T_p(E), \tag{3.12}$$

where $\Delta^h\colon p \mapsto \Delta^h_p \subseteq T_p(E)$ and $\oplus$ is the direct sum sign. The distribution Δ^h is called *horizontal* and symbolically we write $\Delta^v \oplus \Delta^h = T(E)$.

A *vector* at a point $p \in E$ (resp. a *vector field* on E) is said to be *vertical* or *horizontal* if it (resp. its value at p) belongs to Δ^h_p or Δ^v_p, respectively, for the given (resp. any) point p. A vector $Y_p \in T_p(E)$ (resp. vector field $Y \in \mathcal{X}(E)$) is called a *horizontal lift of a vector* $X_{\pi(p)} \in T_{\pi(p)}(M)$ (resp. *vector field* $X \in \mathcal{X}(M)$ on $M = \pi(E)$) if $\pi_*(Y_p) = X_{\pi(p)}$ for the given (resp. any) point $p \in E$. Since $\pi_*|_{\Delta^h_p}\colon \Delta^h_p \to T_{\pi(p)}(M)$ is a vector space isomorphism for all $p \in E$ [23, sec. 1.24], any vector in $T_{\pi(p)}(M)$ (resp. vector field in $\mathcal{X}(M)$) has a unique horizontal lift in $T_p(E)$ (resp. $\mathcal{X}(E)$).

As a result of (3.12), any vector $Y_p \in T_p(E)$ (resp. vector field $Y \in \mathcal{X}(E)$) admits a unique representation $Y_p = Y^v_p \oplus Y^h_p$ (resp. $Y = Y^v \oplus Y^h$) with $Y^v_p \in \Delta^v_p$ and $Y^h_p \in \Delta^h_p$ (resp. $Y^v \in \Delta^v$ and $Y^h \in \Delta^h$). If the distribution $p \mapsto \Delta^h_p$ is differentiable of class C^m, $m \in \mathbb{N} \cup \{0, \infty, \omega\}$, it is said that the *connection* Δ^h *is (differentiable) of class* C^m. A connection Δ^h is of class C^m if and only if, for every C^m vector field Y on E, the vertical Y^v and horizontal Y^h vector fields are of class C^m.

A C^1 *path* $\beta\colon J \to E$ is called *horizontal* (*vertical*) if its tangent vector $\dot\beta$ is horizontal (vertical) vector along β, i.e. $\dot\beta(s) \in \Delta^h_{\beta(s)}$ ($\dot\beta(s) \in \Delta^v_{\beta(s)}$) for all $s \in J$. A *lift* $\bar\gamma\colon J \to E$ of a path $\gamma\colon J \to M$, i.e. $\pi \circ \bar\gamma = \gamma$, is called *horizontal* if $\bar\gamma$ is a horizontal path, i.e. when the vector field $\dot{\bar\gamma}$ tangent to $\bar\gamma$ is horizontal or, equivalently, if $\dot{\bar\gamma}$ is a horizontal lift of $\dot\gamma$. Since $\pi^{-1}(\gamma(J))$ is an $(r+1)$ dimensional submanifold of E, the distribution $p \mapsto \Delta^h_p \cap T_p(\pi^{-1}(\gamma(J)))$ is one-dimensional and, consequently, is integrable. The integral paths of that distribution are horizontal lifts of γ and, *for each* $p \in \pi^{-1}(\gamma(J))$, *there is (locally) a unique horizontal lift* $\bar\gamma_p$ *of* γ *passing through* p. [8]

Definition 3.2. Let $\gamma\colon [\sigma, \tau] \to M$, with $\sigma, \tau \in \mathbb{R}$ and $\sigma \le \tau$, and $\bar\gamma_p$ be the unique horizontal lift of γ in E passing through $p \in \pi^{-1}(\gamma([\sigma, \tau]))$. The *parallel transport (translation, displacement)* generated by (assigned to, defined by) a connection Δ^h is a mapping $\mathsf{P}\colon \gamma \mapsto \mathsf{P}^\gamma$, assigning to the path γ a mapping

$$\mathsf{P}^\gamma\colon \pi^{-1}(\gamma(\sigma)) \to \pi^{-1}(\gamma(\tau)) \qquad \gamma\colon [\sigma, \tau] \to M \tag{3.13}$$

[8] In this sense, a connection Δ^h is an Ehresmann connection [14, p. 314] and *vice versa* [27, pp. 85–89].

such that, for each $p \in \pi^{-1}(\gamma(\sigma))$,

$$\mathsf{P}^{\gamma}(p) := \bar{\gamma}_p(\tau). \tag{3.14}$$

In vector bundles are important the *linear* connections for which is required the parallel transport assigned to them to be linear in a sense that the mapping (3.13) is linear for every path γ (see Subsect. 4.3 below).

Let us now look on the connections Δ^h on a bundle (E, π, M) from a view point of (local) frames and their dual coframes on E. Let $\{e_\mu\}$ be a basis for Δ^h, i.e. $e_\mu \in \Delta^h$ and $\{e_\mu|_p\}$ is a basis for Δ^h_p for all $p \in E$, and $\{e^a\}$ be the coframe for Δ^h, i.e. a collection of 1-forms e^a, $a = n+1, \dots, n+r$, which are linearly independent (relative to functions $E \to \mathbb{K}$) and such that $e^a(X) = 0$ if $X \in \Delta^h$.

Definition 3.3. A frame $\{e_I\}$ in $T(E)$ over E is called *specialized* for a connection Δ^h if the first $n = \dim M$ of its vector fields $\{e_\mu\}$ form a basis for the horizontal distribution Δ^h and its last $r = \dim \pi^{-1}(x)$, $x \in M$, vector fields $\{e_a\}$ form a basis for the vertical distribution Δ^v. Respectively, a coframe $\{e^I\}$ on E is called *specialized* if $\{e^a\}$ is a cobasis for Δ^h and $\{e^\mu\}$ is a cobasis for Δ^v.

The horizontal lifts of vector fields and 1-forms can easily be described in specialized (co)frames. Indeed, let $\{e_I\}$ and $\{e^I\}$ be respectively a specialized frame and its dual coframe. Define a frame $\{E_\mu\}$ and its dual coframe $\{E^\mu\}$ on M which are π-related to $\{e_I\}$ and $\{e^I\}$, i.e. $E_\mu := \pi_*(e_\mu)$ and $e^\mu := \pi^*(E^\mu) = E^\mu \circ \pi_*$. [9] If $Y = Y^\mu E_\mu \in \mathcal{X}(M)$ and $\phi = \phi_\mu e^\mu \in \Lambda^1(M)$, then their horizontal lifts (from M to E) respectively are

$$\bar{Y} = (Y^\mu \circ \pi) e_\mu \qquad \bar{\phi} = (\phi_\mu \circ \pi) e^\mu. \tag{3.15}$$

The specialized (co)frames transform into each other according to the general rules (3.4) in which the transformation matrix and its inverse have the following block structure:

$$A = \begin{pmatrix} [A^\nu_\mu] & 0_{n\times r} \\ 0_{r\times n} & [A^b_a] \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} [A^\nu_\mu]^{-1} & 0_{n\times r} \\ 0_{r\times n} & [A^b_a]^{-1} \end{pmatrix}, \tag{3.16}$$

where $A^\nu_\mu, A^b_a \colon E \to \mathbb{K}$ and the functions A^ν_μ are constant on the fibres of the bundle (E, π, M), that is, we have

$$A^\nu_\mu = B^\nu_\mu \circ \pi \quad \text{or} \quad \frac{\partial A^\nu_\mu}{\partial u^a} = 0 \tag{3.17}$$

for some nondegenerate matrix-valued function $[B^\nu_\mu]$ on M. Besides, in a case of vector bundle, the functions A^a_b are also constant on the fibres of the bundle (E, π, M), i.e.

$$A^b_a = B^b_a \circ \pi \quad \text{or} \quad \frac{\partial A^b_a}{\partial u^a} = 0 \tag{3.18}$$

for some nondegenerate matrix-valued function $B = [B^b_a]$ on M. Changes like (3.4), with A given by (3.16), respect the fibre as well as the connection structure of the bundle.

Let E be a C^2 manifold and Δ^h a C^1 connection on (E, π, M). The components C^K_{IJ} of the anholonomy object of a *specialized* frame $\{e_I\}$ are (local) functions on E defined by (see (2.6))

$$[e_I, e_J]_{\_} =: C^K_{IJ} e_K \tag{3.19}$$

[9] Recall, $\pi_*|_{\Delta^h_p} \colon \Delta^h_p \to T_{\pi(p)}(M)$ is a vector space isomorphism.

and are naturally divided into the following six groups (cf. [20, p. 21]:

$$\{C^{\lambda}_{\mu\nu}\}, \quad \{C^{a}_{\mu\nu}\}, \quad \{C^{\lambda}_{\mu b}=0\}, \quad \{C^{\lambda}_{ab}=0\}, \quad \{C^{c}_{\mu b}\}, \quad \{C^{c}_{ab}\}. \tag{3.20}$$

The functions $C^{\lambda}_{\mu\nu}$ are constant on the fibres of (E,π,M), precisely $C^{\lambda}_{\mu\nu}=f^{\lambda}_{\mu\nu}\circ\pi$ where $f^{\lambda}_{\mu\nu}$ are the components of the anholonomy object for the π-related frame $\{\pi_*(e_\mu)\}$ on M, as the commutators of π-related vector fields are π-related [5, sec. 1.55]. Besides, since the vertical distribution Δ^v is integrable (the space Δ^v_p is the space tangent to the fibre through $p\in E$ at p), we have

$$[e_a,e_b]_- = C^{c}_{ab}e_c \tag{3.21}$$

(so that $C^{\lambda}_{ab}=0$), due to which C^{c}_{ab} are called components of the vertical anholonomy object. To prove that $C^{\lambda}_{\mu b}=0$, one should expand $\{e_I\}$ along $\{\partial_I=\frac{\partial}{\partial u^I}\}$, with $\{u^I\}$ being some bundle coordinates, viz. $e_\mu=e^{\nu}_{\mu}\partial_\nu+e^{b}_{\mu}\partial_b$ and $e_a=e^{b}_{a}\partial_b$, with some functions e^{ν}_{μ}, e^{b}_{μ} and e^{b}_{a}, and to notice that e^{ν}_{μ} are constant on the fibres, i.e. $\partial_a(e^{\nu}_{\mu})=0$.

The non-trivial mixed "vertical-horizontal" components between (3.20), viz. $C^{a}_{\mu\nu}$ and $C^{a}_{\mu b}$, are important characteristics of the connection Δ^h. The functions

$$^{\circ}\Gamma^{a}_{b\mu} := +C^{a}_{b\mu} = -C^{a}_{\mu b} \tag{3.22a}$$
$$R^{a}_{\mu\nu} := +C^{a}_{\mu\nu} = -C^{a}_{\nu\mu}, \tag{3.22b}$$

which enter into the commutators

$$\mathcal{L}_{e_\mu}e_b = [e_\mu,e_b]_- = {}^{\circ}\Gamma^{a}_{b\mu}e_a \tag{3.23a}$$
$$[e_\mu,e_\nu]_- = R^{a}_{\mu\nu}e_a + C^{\lambda}_{\mu\nu}e_\lambda, \tag{3.23b}$$

are called respectively the *fibre coefficients of* Δ^h (or *components of the connection object of* Δ^h) and *fibre components of the curvature of* Δ^h (or *components of the curvature (object) of* Δ^h) in $\{e_I\}$. Under a change (3.4), with a matrix (3.16), of the specialized frame, the functions (3.22) transform into respectively

$$^{\circ}\tilde{\Gamma}^{a}_{b\mu} = A^{\nu}_{\mu}\big([A^{f}_{e}]^{-1}\big)^{a}_{d}\big({}^{\circ}\Gamma^{d}_{c\nu}A^{c}_{b}+e_\nu(A^{d}_{b})\big) \tag{3.24a}$$
$$\tilde{R}^{a}_{\mu\nu} = \big([A^{f}_{e}]^{-1}\big)^{a}_{b}A^{\lambda}_{\mu}A^{\varrho}_{\nu}R^{b}_{\lambda\varrho}, \tag{3.24b}$$

which formulae are direct consequences of (3.23). If we put $\bar{A}:=[A^{b}_{a}]$, ${}^{\circ}\Gamma_\nu := [{}^{\circ}\Gamma^{d}_{c\nu}]$, and ${}^{\circ}\tilde{\Gamma}_\nu := [{}^{\circ}\tilde{\Gamma}^{d}_{c\nu}]$, then (3.24a) is tantamount to

$$\begin{aligned} {}^{\circ}\tilde{\Gamma}_\mu &= A^{\nu}_{\mu}\bar{A}^{-1}\cdot({}^{\circ}\Gamma_\nu\cdot\bar{A}+e_\nu(\bar{A})) \\ &= A^{\nu}_{\mu}(\bar{A}^{-1}\cdot{}^{\circ}\Gamma_\nu - e_\nu(\bar{A}^{-1}))\cdot\bar{A}. \end{aligned} \tag{3.25}$$

Up to a meaning of the matrices $[A^{\nu}_{\mu}]$ and $\bar{A}$ and the size of the matrices ${}^{\circ}\Gamma_\nu$ and $\bar{A}$, the last equation is identical with the one expressing the transformed matrices of the coefficients of a linear connection (covariant derivative operator) in a vector bundle [28, eq. (3.5)] on which we shall return later in this work (see Sect. 4, in particular equation (4.32′) in it). Equation (3.24b) indicates that $R^{a}_{\mu\nu}$ are components of a tensor, viz.

$$\Omega := \frac{1}{2}R^{a}_{\mu\nu}e_a\otimes e^\mu\wedge e^\nu, \tag{3.26}$$

called *curvature tensor* of the connection Δ^h. By (3.23a), the horizontal distribution Δ^h is (locally) integrable iff its curvature tensor vanishes, $\Omega=0$.

Definition 3.4. A connection with vanishing curvature tensor is called *flat*, or *integrable*, or *curvature free*.

Proposition 3.1. *The flat connections are the only ones that may admit holonomic specialized frames.*

Proof. See definition 3.4 and (3.23b). □

The above considerations of specialized (co)frames for a connection Δ^h on a bundle (E, π, M) were *global* as we supposed that these (co)frames are defined on the whole bundle space E, which is always possible if no smoothness conditions on Δ^h are imposed. Below we shall show how *local* specialized (co)frames can be defined via local bundle coordinates on E.

Let $\{u^I\}$ be local bundle coordinates on an open set $U \subseteq E$. They define on $T(U) \subseteq T(E)$ the local basis $\{\partial_I := \frac{\partial}{\partial u^I}\}$, so that any vector can be expended along its vectors. In particular, we can do so with any basic vector field e_I^U of a *specialized* frame $\{e_I\}$ restricted to U, $e_I^U := e_I|_U$. Since $\{\partial_a|_p\}$, with $p \in U$, is a basis for Δ_p^v, we can write

$$(e_\mu^U, e_a^U) = (A_\mu^\nu \partial_\mu + A_\mu^a \partial_a, A_a^b \partial_b) = (\partial_\nu, \partial_b) \cdot \begin{pmatrix} [A_\mu^\nu] & 0 \\ [A_\mu^b] & [A_a^b] \end{pmatrix}, \tag{3.27}$$

where $[A_\mu^\nu]$ and $[A_a^b]$ are non-degenerate matrix-valued functions on U. [10]

Definition 3.5. A frame $\{X_I\}$ over U in $T(U)$ is called *adapted (to the coordinates $\{u^I\}$ in U)* for a connection Δ^h if it is the specialized frame obtained from (3.27) via admissible transformation (3.4) with the matrix $A = \begin{pmatrix} [A_\mu^\nu]^{-1} & 0 \\ 0 & [A_a^b]^{-1} \end{pmatrix}$.

Exercise 3.1. An arbitrary *specialized* frame $\{e_I^U\}$ in $T(E)$ over U enters in the definition of a frame $\{X_I\}$ adapted to bundle coordinates $\{u^I\}$ on U. Prove that $\{X_I\}$ is independent of the particular choice of the frame $\{e_I^U\}$. (Hint: apply definition 3.5 and (3.4a) with A given by (3.16).) The below-derived equality (3.34) is an indirect proof of that fact too.

According to (3.4) and definition 3.5, the adapted frame $\{X_I\}$ and the corresponding to it *adapted coframe* $\{\omega^I\}$ are given by

$$X_\mu = \partial_\mu + \Gamma_\mu^a \partial_a \qquad X_a = \partial_a \tag{3.28a}$$

$$\omega^\mu = \mathrm{d}u^\mu \qquad \omega^a = \mathrm{d}u^a - \Gamma_\mu^a \mathrm{d}u^\mu. \tag{3.28b}$$

Here the functions $\Gamma_\mu^a \colon U \to \mathbb{K}$ are defined via

$$[\Gamma_\mu^a] = +[A_\nu^a] \cdot [A_\mu^\nu]^{-1} \tag{3.29}$$

and are called *(2-index) coefficients* of Δ^h. In a matrix form, the equations (3.28) can be written as

$$(X_\mu, X_a) = (\partial_\nu, \partial_b) \cdot \begin{bmatrix} \delta_\mu^\nu & 0 \\ +\Gamma_\mu^b & \delta_a^b \end{bmatrix} \quad \begin{pmatrix} \omega^\mu \\ \omega^a \end{pmatrix} = \begin{bmatrix} \delta_\nu^\mu & 0 \\ -\Gamma_\nu^a & \delta_b^a \end{bmatrix} \cdot \begin{pmatrix} \mathrm{d}u^\nu \\ \mathrm{d}u^b \end{pmatrix}. \tag{3.30}$$

The operators $X_\mu = \partial_\mu + \Gamma_\mu^a \partial_a$ are known as *covariant derivatives on* $T(U)$ and the plus sing in (3.28a) before Γ_μ^a (hence in the r.h.s. of (3.29)) is conventional.

If $\{u^I\}$ and $\{\tilde{u}^I\}$ are local coordinates on open sets $U \subseteq E$ and $\tilde{U} \subseteq E$, respectively, and $U \cap \tilde{U} \neq \varnothing$, then, on the overlapping set $U \cap \tilde{U}$, a problem arises: how are connected the adapted frames corresponding to these coordinates? Let us mark with a tilde all quantities

[10] The non-degeneracy of $[A_\mu^\nu]$ follows from the fact that the vector fields $\pi_*|_{\Delta^h}(e_\mu^U) = A_\mu^\nu \pi_*(\frac{\partial}{\partial u^\mu})$ form a basis for $\mathcal{X}(\pi(U)) \subseteq \mathcal{X}(M)$.

that refer to the coordinates $\{\tilde{u}^I\}$. Since the adapted frames are, by definitions, specialized ones, we can write (see (3.4))

$$(\tilde{X}_\mu, \tilde{X}_a) = (X_\nu, X_b) \cdot A \qquad \begin{pmatrix} \tilde{\omega}^\mu \\ \tilde{\omega}^a \end{pmatrix} = A^{-1} \cdot \begin{pmatrix} \omega^\nu \\ \omega^b \end{pmatrix}, \tag{3.31a}$$

where the transformation matrix A and its inverse have the form (3.16). Recalling (3.2) and (3.3), from these equalities, we get

$$A = \operatorname{diag}\Big(\Big[\frac{\partial u^\nu}{\partial \tilde{u}^\mu}\Big], \Big[\frac{\partial u^b}{\partial \tilde{u}^a}\Big]\Big) = \begin{pmatrix} \big[\frac{\partial u^\nu}{\partial \tilde{u}^\mu}\big] & 0 \\ 0 & \big[\frac{\partial u^b}{\partial \tilde{u}^a}\big] \end{pmatrix} . \tag{3.31b}$$

Combining (3.29) and (3.31), one can easily prove

Proposition 3.2. *A change $\{u^I\} \mapsto \{\tilde{u}^I\}$ of the local bundle coordinates implies the following transformation of the 2-index coefficients of the connection:*

$$\Gamma_\mu^a \mapsto \tilde{\Gamma}_\mu^a = \Big(\frac{\partial \tilde{u}^a}{\partial u^b}\Gamma_\nu^b + \frac{\partial \tilde{u}^a}{\partial u^\nu}\Big)\frac{\partial u^\nu}{\partial \tilde{u}^\mu}. \tag{3.32}$$

It is obvious, a connection Δ^h is of class C^m, $m \in \mathbb{N} \cup \{0\}$, if and only if its coefficients Γ_μ^a are C^m functions on U, provided ∂_I are C^m vector fields on U (which is the case when E is a C^{m+1} manifold). By virtue of (3.32), the C^{m+1} changes of the local bundle coordinates preserve the C^m differentiability of Γ_μ^a. Thus the C^{m+1} differentiability of the base M and bundle E spaces is a necessary condition for existence of C^m connections on (E, π, M); as we assumed $m = 1$ in this work, the connections considered here can be at most of differentiability class C^1.

The next proposition states that a connection on a bundle is locally equivalent to a geometric object whose components transform like (3.32).

Proposition 3.3. *To any connection Δ^h in a bundle (E, π, M) can be assigned, according to the procedure described above, a geometrical object on E whose components Γ_μ^a in bundle coordinates $\{u^I\}$ on E transform according to* (3.32) *under a change $\{u^I\} \mapsto \{\tilde{u}^I\}$ of the bundle coordinates on the intersection of the domains of $\{u^I\}$ and $\{\tilde{u}^I\}$. Conversely, given a geometrical object on E with local transformation law* (3.32)*, there is a unique connection Δ^h in (E, π, M) which generates the components of that object as described above.*

Proof. The first part of the statement was proved above, when we constructed the adapted frame (3.28a) and derived (3.32). To prove the second part, choose a point $p \in E$ and some local coordinates $\{u^I\}$ on an open set U in E containing p in which the geometrical object mentioned has local components Γ_μ^a. Define a local frame $\{X_I\} = \{X_\mu, X_a\}$ on U by (3.28a). The required connection is then $\Delta^h\colon q \mapsto \Delta_q^h := \{r^\mu X_\mu|_q : r^\mu \in \mathbb{K}\}$ for any $q \in U$, which means that Δ_q^h is the linear cover of $\{X_\mu|_q\}$. The transformation law (3.32) insures the independence of Δ^h from the local coordinates employed in its definition. □

From the construction of an adapted frame $\{X_I\}$, as well as from the proof of proposition 3.3, follows that the set of vectors $\{X_\mu\}$ is a basis for the horizontal distribution Δ^h and the set $\{X_a\}$ is a basis for the vertical distribution Δ^v. The matrix of the restricted tangent projection $\pi_*|_{\Delta^h}$ in bundle coordinates $\{u^\mu = x^\mu \circ \pi, u^a\}$ on E, where $\{x^\mu\}$ are local coordinates on M, is the identity matrix as $(\pi_*|_{\Delta_p^h})_\mu^\nu = \frac{\partial(x^\mu \circ \pi)}{\partial u^\mu}\Big|_p = \delta_\mu^\nu$ for any point p in the domain of $\{u^I\}$. Hereof we get

$$\pi_*|_{\Delta^h}(X_\mu) = \frac{\partial}{\partial x^\mu} \qquad \Big(\iff \pi_*|_{\Delta_p^h}(X_\mu|_p) = \frac{\partial}{\partial x^\mu}\Big|_{\pi(p)}\Big). \tag{3.33}$$

In particular, from here follows that $\pi_*|_{\Delta_p^h} : \Delta_p^h \to T_{\pi(p)}(M)$ is a vector space isomorphism. The inverse to equation (3.33), viz.

$$X_\mu = (\pi_*|_{\Delta^h})^{-1}\Big(\frac{\partial}{\partial x^\mu}\Big) = (\pi_*|_{\Delta^h})^{-1} \circ \pi_*\Big(\frac{\partial}{\partial u^\mu}\Big), \tag{3.34}$$

can be used in an equivalent definition of a frame $\{X_I\}$ adapted to local coordinates $\{u^I\}$, namely, this is the frame $\big((\pi_*|_{\Delta^h})^{-1} \circ \pi_*\big(\frac{\partial}{\partial u^\mu}\big), \frac{\partial}{\partial u^a}\big)$. If one accepts such a definition of an adapted frame for Δ^h, the (2-index) coefficients of Δ^h have to be defined via the expansion (3.28a); the only changes this may entail are in the proofs of some results, like (3.31) and (3.32).

It is useful to be recorded also the simple fact that, by construction, we have

$$\pi_*(X_a) = 0. \tag{3.35}$$

Let E be a C^2 manifold and Δ^h be a C^1 connection. The *adapted frames are generally anholonomic* as the commutators between the basic vector fields of the adapted frame (3.28a) are (cf. (3.20) and (3.22))

$$[X_\mu, X_\nu]_{\_} = R^a_{\mu\nu} X_a \quad [X_\mu, X_b]_{\_} = {}^\circ\Gamma^a_{b\mu} X_a \quad [X_a, X_b]_{\_} = 0, \tag{3.36}$$

with

$$R^a_{\mu\nu} = \partial_\mu(\Gamma^a_\nu) - \partial_\nu(\Gamma^a_\mu) + \Gamma^b_\mu \partial_b(\Gamma^a_\nu) - \Gamma^b_\nu \partial_b(\Gamma^a_\mu) = X_\mu(\Gamma^a_\nu) - X_\nu(\Gamma^a_\mu) \tag{3.37a}$$

$${}^\circ\Gamma^a_{b\mu} = -\partial_b(\Gamma^a_\mu) = -X_b(\Gamma^a_\mu) \tag{3.37b}$$

being the fibre components of the curvature and fibre coefficients of the connection. An obvious corollary from (3.37) is

Proposition 3.4. *An adapted frame is holonomic if and only if*

$$R^a_{\mu\nu} = 0 \quad (\iff \quad \Omega = 0) \quad {}^\circ\Gamma^a_{b\mu} = 0. \tag{3.38}$$

Therefore only the flat (integrable) C^1 connections, for which $\Omega = 0$, may admit holonomic adapted frames. Besides, as a consequence of (3.37b) and (3.38), such connections admit holonomic adapted frames on $U \subseteq E$ if and only if there are local coordinates on U in which the coefficients Γ^a_μ are constant on the fibres passing through U, i.e. iff $\Gamma^a_\mu = G^a_\mu \circ \pi$ for some functions $G^a_\mu : \pi(U) \to \mathbb{K}$, which is equivalent to $\partial_b(\Gamma^a_\mu) = 0$.

Example 3.1 (horizontal lifting of a path)**.** Recall, the horizontal lift of a C^1 path $\gamma : J \to M$ passing through a point $p \in \pi^{-1}(\gamma(t_0))$ for some $t_0 \in J$ is the unique path $\bar\gamma_p : J \to E$ such that $\pi \circ \bar\gamma_p = \gamma$, $\bar\gamma_p(t_0) = p$, and $\dot{\bar\gamma}_p(t) \in \Delta^h_{\bar\gamma_p(t)}$ for all $t \in J$. As in a specialized frame $\{e_I\}$ the relation $X_p \in \Delta^h_p$ is equivalent to $e^a(X) = 0$ for any $X \in \mathcal{X}(M)$, in an adapted coframe, given by (3.28b), the horizontal lift $\bar\gamma_p$ of γ is the unique solution of the initial value problem

$$\omega^a(\dot{\bar\gamma}_p) = 0 \tag{3.39a}$$

$$\bar\gamma_p(t_0) = p \tag{3.39b}$$

which is tantamount to any one of the initial-value problems ($t \in J$)

$$\dot{\bar\gamma}^a_p(t) - \Gamma^a_\mu(\bar\gamma_p(t))\dot{\bar\gamma}^\mu_p(t) = 0 \tag{3.39$'$a}$$

$$\bar\gamma^I_p(t_0) = p^I := u^I(p) \tag{3.39$'$b}$$

$$\frac{\mathrm{d}(u^a \circ \bar{\gamma}_p(t))}{\mathrm{d}t} - \Gamma^a_\mu(\bar{\gamma}_p(t)) \frac{\mathrm{d}(x^\mu \circ \gamma(t))}{\mathrm{d}t} = 0 \tag{3.39$''$a}$$

$$u^I(\bar{\gamma}_p(t_0)) = u^I(p), \tag{3.39$''$b}$$

where $\{x^\mu\}$ are the local coordinates in the base that induce the basic coordinates $\{u^\mu\}$ on the bundle space, $u^\mu = x^\mu \circ \pi$. (Note that the quantities $\frac{\mathrm{d}(x^\mu \circ \gamma(t))}{\mathrm{d}t}$, entering into (3.39$''$a), are the components of the vector $\dot{\gamma}$ tangent to γ at parameter value t.) One may call (3.39a), or any one of its versions (3.39$'$a) or (3.39$''$a), the *parallel transport equation* in an adapted frame as it uniquely determines the parallel transport along the restriction of γ to any closed subinterval in J (see definition 3.2).

Example 3.2 (the equation of geodesic paths)**.** A connection Δ^h on the tangent bundle $(T(M), \pi_T, M)$ of a manifold M is called a *connection on* M. In this case, equation (3.39) defines also the geodesics (relative to Δ^h) in M. A C^2 path $\gamma\colon J \to M$ in a C^2 manifold M is called a *geodesic path* if its tangent vector field $\dot{\gamma}$ undergoes parallel transport along the same path γ, i.e. $\mathsf{P}^{\gamma|[\sigma,\tau]}(\dot{\gamma}(\sigma)) = \dot{\gamma}(\tau)$ for all $\sigma, \tau \in J$, which means that the lifted path $\dot{\gamma}\colon J \to T(M)$ is a *horizontal* lift of γ (relative to Δ^h). So, if $\{x^\mu\}$ are local coordinates on $\pi(U) \in M$ and the bundle coordinates on $U \subseteq E$ are such that [5, sect. 1.25] $u^\mu = x^\mu \circ \pi$ and $u^{n+\mu} = \mathrm{d}x^\mu$ $(\mu, \nu, \dots = 1, \dots, n = \dim M)$, then (3.39$''$a) transforms into the *geodesic equation* (on M)

$$\frac{\mathrm{d}^2(x^\mu \circ \gamma(t))}{\mathrm{d}t^2} - \Gamma^{n+\mu}_\nu(\dot{\gamma}(t)) \frac{\mathrm{d}(x^\nu \circ \gamma(t))}{\mathrm{d}t} = 0 \qquad t \in J, \tag{3.40}$$

which (locally) defines all geodesics in M. (With obvious renumbering of the indices, one usually writes Γ^μ_ν for $\Gamma^{n+\mu}_\nu$.) Of course, a particular geodesic is specified by fixing some initial values for $\gamma(t_0)$ and $\dot{\gamma}(t_0)$ for some $t_o \in J$. Notice, equation (3.40) is an equation for a path γ in M, while (3.39$''$a) is an equation for the lifted path $\bar{\gamma}$ in $T(M)$ provided the path γ in M is known; for a geodesic path, evidently, we have $\bar{\gamma} = \dot{\gamma}$.

4. Connections on vector bundles

In this section, by (E, π, M) we shall denote an arbitrary *vector* bundle [23]. A peculiarity of such bundles is that their fibres are isomorphic vector spaces, which leads to a natural description of the vertical distribution Δ^v on their fibre spaces, as well as to existence of a kind of differentiation of their sections (known as covariant differentiation).

In the vector bundles are used, as we shall do in this section, the so-called vector bundle coordinates which are linear on their fibres and are constructed as follows (cf. [27, p. 30]).

Let $\{e_a\}$ be a frame in E over a subset $U_M \subseteq M$, i.e. $\{e_a(x)\}$ to be a basis in $\pi^{-1}(x)$ for all $x \in U_M$. Then, for each $p \in \pi^{-1}(U_M)$, we have a unique expansion $p = p^a e_a(\pi(p))$ for some numbers $p^a \in \mathbb{K}$. The *vector fibre coordinates* $\{u^a\}$ on $\pi^{-1}(U_M)$ induced (generated) by the frame $\{e_a\}$ are defined via $u^a(p) := p^a$ and hence can be identified with the elements of the coframe $\{e^a\}$ dual to $\{e_a\}$, i.e. $u^a = e^a$. Conversely, if $\{u^I\}$ are coordinates on $\pi^{-1}(U_M)$ for some $U_M \subseteq M$ which are linear on the fibres over U_M, then there is a unique frame $\{e_a\}$ in $\pi^{-1}(U_M)$ which generates $\{u^a\}$ as just described; indeed, considering $u^{n+1}, \dots, u^{n+r}$ as 1-forms on $\pi^{-1}(U_M)$, one should define the frame $\{e_a\}$ required as a one whose dual is $\{u^a\}$, i.e. via the conditions $u^a(e_b) = \delta^a_b$.

A collection $\{u^I\}$ of basic coordinates $\{u^\mu\}$ and vector fibre coordinates $\{u^a\}$ on $\pi^{-1}(U_M)$ is called *vector bundle coordinates* on $\pi^{-1}(U_M)$. Only such coordinates on E will be employed in this section.

4.1. Vertical lifts

The idea of describing the vertical distribution Δ^v on a vector bundle is that, if L is a vector space, then to any $Y \in L$ there corresponds a 'vertical' vector field $Y^v \in \mathcal{X}(L) = \mathrm{Sec}(T(L), \pi_T, L)$ whose value at $X \in L$ is the vector tangent to the path $t \mapsto X + tY \in L$, with $t \in \mathbb{R}$, at $t = 0$, i.e. $Y^v|_X := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(X + tY)$. Here and below, with $\mathrm{Sec}(E, \pi, M)$ (resp. $\mathrm{Sec}^m(E, \pi, M)$ with $m \in \mathbb{N} \cup \{0\}$) we denote the module of sections (resp. C^m sections) of a bundle (E, π, M) (resp. of a C^{m+1} bundle (E, π, M)).

Let (E, π, M) be a vector bundle and Δ^v the vertical distribution on it, viz., for each $p \in E$, $\Delta^v \colon p \mapsto \Delta^v_p := T_p(\pi^{-1}(\pi(p)))$. To every $Y \in \mathrm{Sec}(E, \pi, M)$, we assign a *vertical* vector field $Y^v \in \Delta^v$ on E such that, for $p \in E$,

$$Y^v_p := Y^v|_p := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(p + tY|_{\pi(p)}). \tag{4.1}$$

(The mapping $(p, Y_{\pi(p)}) \mapsto Y^v_p$ defines an isomorphism from the pullback bundle π^*E into the vertical bundle $\mathcal{V}(E)$ — see [23, sections 1.27 and 1.28] and also [27, p. 41, exercises 2.2.1 and 2.2.2].)

Lemma 4.1. *Let $\{u^a\}$ be vector fibre coordinates generated by a frame $\{e_a\}$ on M. If $Y \in \mathrm{Sec}(E, \pi, M)$ and $Y = Y^a e_a$, then*

$$Y^v = (Y^a \circ \pi)\frac{\partial}{\partial u^a}. \tag{4.2}$$

Proof. Using the definition (4.1), we get for $p \in E$:

$$\begin{aligned} Y^v|_p &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(p + tY|_{\pi(p)}) = \frac{\mathrm{d}(u^a(p + tY|_{\pi(p)}))}{\mathrm{d}t}\Big|_{t=0}\frac{\partial}{\partial u^a}\Big|_p \\ &= \frac{\mathrm{d}(p^a + tY^a(\pi(p)))}{\mathrm{d}t}\Big|_{t=0}\frac{\partial}{\partial u^a}\Big|_p = Y^a(\pi(p))\frac{\partial}{\partial u^a}\Big|_p = \Big((Y^a \circ \pi)\cdot\frac{\partial}{\partial u^a}\Big)\Big|_p. \end{aligned}$$

□

If $Y \in \mathrm{Sec}(E, \pi, M)$, the vector field $Y^v := v(Y) \in \Delta^v$, defined via (4.1), is called the *vertical lift of the section* Y and, in vector bundle coordinates, is (locally) given by (4.2).

Proposition 4.1. *The mapping*

$$\begin{aligned} &v\colon \mathrm{Sec}(E, \pi, M) \to \{\text{vector fields in } \Delta^v\} \\ &v\colon Y \mapsto Y^v\colon p \mapsto Y^v|_p := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(p + tY_{\pi(p)}) \end{aligned} \tag{4.3}$$

is a bijection and it and its inverse are linear mapping.

Proof. The linearity and injectivity of v follow directly from (4.1). Now we shall prove that, for each $Z \in \Delta^v$, there is a $Y \in \mathrm{Sec}(E, \pi, M)$ such that $Y^v = Z$, i.e. v is also surjective. Let $Z = Z^a\frac{\partial}{\partial u^a}$, with $\{u^I\}$ being (local) vector bundle coordinates on E and the functions Z^a being constant on the fibres of E, that is $Z^I = z^I \circ \pi$ for some functions z^I on M. Define $Y = z^a e_a$ with $\{e_a\}$ being the frame in E over M generating $\{u^I\}$. By lemma 4.1, we have $Y^v = (z^a \circ \pi)\frac{\partial}{\partial u^a} = Z^a\frac{\partial}{\partial u^a} = Z$. The linearity of v^{-1} follows from here too. □

Consider a section ω of the bundle dual to (E, π, M) [23]. Its *vertical lift* ω_v is a 1-form on Δ^v such that, for $Z \in \Delta^v$ and $p \in E$, $\omega_v(Z)|_p = \omega(Y)|_{\pi(p)}$ for the unique section $Y \in \mathrm{Sec}(E, \pi, M)$ with $Y^v = Z$ (see proposition 4.1), i.e. we have $\omega_v(\cdot)|_p = (\omega \circ v^{-1}(\cdot))|_{\pi(p)}$ which means that

$$\omega_v(Z) = (\omega \circ v^{-1}(Z)) \circ \pi \quad \text{or} \quad \omega_v(Y^v)|_p = \omega(Y)|_{\pi(p)} \quad (= \omega_{\pi(p)}(Y_{\pi(p)})). \tag{4.4}$$

If $\{e^a = u^a\}$ is the coframe dual to $\{e_a\}$, and $\omega = \omega_a e^a$, then in the coframe $\{\mathrm{d}u^a\}$ dual to $\{\frac{\partial}{\partial u^a}\}$, we can write (cf. (4.2))

$$\omega_v = (\omega_a \circ \pi)\mathrm{d}u^a. \tag{4.5}$$

It should be mentioned, 'vertical' lifts of vector fields or 1-forms over the base space are generally not defined unless $E = T(M)$ or $E = T^*(M)$, respectively. [11]

Let Δ^h be a connection on (E, π, M) and $\varphi\colon E \to \mathbb{K}$ be a C^1 mapping. Since any $X \in \mathcal{X}(E)$ can uniquely be written as a direct sum $X = v(X) \oplus h(X)$, with $v(X) \in \Delta^v$ and $h(X) \in \Delta^h$, we have $\varphi_*(X) = \varphi_*(v(X)) + \varphi_*(h(X)) \in \mathcal{X}(M)$. If $\{Z_I\}$ is a specialized frame in $T(E)$ and $\{Z^I\}$ is its dual coframe of one-forms on $\mathcal{X}(E)$, we immediately get

$$\varphi_* = (\varphi_*(Z_a))Z^a + (\varphi_*(Z_\mu))Z^\mu = (Z_a(\varphi))Z^a + (Z_\mu(\varphi))Z^\mu \tag{4.6}$$

as $X = X^I Z_I$ entails $v(X) = X^a Z_a$ and $h(X) = X^\mu Z_\mu$; in particular, (4.6) holds in any adapted (co)frame (3.28) and/or a section φ of the bundle (E, π, M). If $\{u^I\}$ are vector bundle coordinates, in the (co)frame (3.28) adapted to them, we have $Z_\mu = X_\mu$, $Z_a = \partial_a$, $Z^\mu = \omega^\mu = \mathrm{d}u^\mu$, $Z^a = \omega^a$, and we can write the expansion $\varphi = \varphi_a u^a$ with $\varphi_a\colon E \to \mathbb{K}$. Thus (4.6) takes the form

$$\varphi_* = \varphi_a \omega^a + (X_\mu(\varphi_a u^a))\omega^\mu = \varphi_v + (X_\mu(\varphi_a u^a))\omega^\mu,$$

where (4.5) was applied.

A *section* Y *of* (E, π, M) and *section* ω *of the bundle dual to* (E, π, M) can be lifted *vertically* via the mappings

$$v\colon Y \mapsto Y^v \in \Delta^v \tag{4.7a}$$

$$\omega \mapsto \omega_v \tag{4.7b}$$

respectively given by (4.3) and (4.4) (see also (4.2) and (4.5)). These mappings do not require a connection and arise only from the fibre structure of the bundle space induced from the projection $\pi\colon E \to M$.

If a connection Δ^h on (E, π, M) is given, it generates *horizontal lifts of the vector fields on the base space* M and of the *one-forms on the same base space* M into respectively vector fields in Δ^h and linear mappings on the vector fields in Δ^h. Precisely, if $F \in \mathcal{X}(M)$ and $\phi \in \Lambda^1(M)$, their *horizontal lifts* are defined by the mappings [12]

$$F \mapsto F^h \in \Delta^h \quad \text{with} \quad F^h\colon p \mapsto F^h_p := (\pi_*|_{\Delta^h_p})^{-1}(F_{\pi(p)}) \qquad p \in E \tag{4.8a}$$

$$\phi \mapsto \phi_h \quad \text{with} \quad \phi_h := \phi \circ \pi_*|_{\Delta^h}\colon p \mapsto \phi_h|_p = \phi|_{\pi(p)} \circ (\pi_*|_{\Delta^h_p}). \tag{4.8b}$$

The horizontal lift ϕ_h of ϕ can also be defined alternatively via

$$\phi_h(F^h)|_p = \phi(F)|_{\pi(p)} \tag{4.9}$$

which equation is tantamount to (4.8b).

Let $\{u^\mu = x^\mu \circ \pi, u^a\}$ be vector bundle coordinates and $\{X_I\}$ (resp. $\{\omega_I\}$) be the adapted to them frame (resp. coframe) constructed from them according to (3.28). If $Y = Y^a e_a$,

[11] Since $\pi_*(\Delta^v_p) = 0_{\pi(p)} \in T_{\pi(p)}(M)$, $p \in E$, we can say that only the zero vector field over M has vertical lifts relative to π and any vector field in Δ^v is its vertical lift. This conclusion is independent of the existence of a connection on (E, π, M) and depends only on the fibre structure of E induced by π.

[12] Alternatively, one may define $\phi'_h = \phi \circ \pi_* = \pi^*(\phi)$, which expands the domain of ϕ_h, defined by (4.8b), on the whole space $\mathcal{X}(E)$. Obviously, $\phi'_h(Z) = \phi_h(Z)$ for $Z \in \Delta^h \subseteq \mathcal{X}(E)$ and $\phi'_h(Z) = 0$ for $Z \in \mathcal{X}(E) \setminus \{X \in \Delta^h\}$.

$\omega = \omega_a e^a$, $F = F^\mu \frac{\partial}{\partial x^\mu} \in \mathcal{X}(M)$, and $\phi = \phi_\mu \mathrm{d}x^\mu \in \Lambda^1(M)$, the equations (4.2) and (4.5) imply

$$Y^v = (Y^a \circ \pi) X_a \quad \omega_v = (\omega_a \circ \pi)\omega^a, \tag{4.10}$$

while from (4.8) and (3.33), one gets

$$F^h = (F^\mu \circ \pi) X_\mu \quad \phi_h = (\phi_\mu \circ \pi)\omega^\mu, \tag{4.11}$$

which agree with (3.15).

4.2. The tangent and cotangent bundle cases

As an example, in the present subsection is considered a connection Δ^h on the tangent bundle $(T(M), \pi_T, M)$ over a manifold M.

A vector field $Y \in \mathcal{X}(M) = \mathrm{Sec}(T(M), \pi_T, M)$ has unique vertical lift $Y^v \in \Delta^v$ (which is independent of Δ^h) and unique *horizontal* lift given by (see (4.3))

$$Y^v := v(Y) \in \Delta^v \quad Y^h := ((\pi_T)_*|_{\Delta^h})^{-1}(Y) \in \Delta^h, \tag{4.12}$$

the last equality meaning that $Y^h_p := ((\pi_T)_*|_{\Delta^h_p})^{-1}(Y_p)$, which is correct as $(\pi_T)_*|_{\Delta^h_p} \colon \Delta^h_p \to T_{\pi(p)}(M)$ is an isomorphism. Respectively, if ω is 1-form on M, it has vertical lift ω_v (which is independent of Δ^h) and *horizontal* lift ω_h, which is one-form on Δ^h, defined by (see (4.4))

$$\omega_v(Z) = (\omega \circ v^{-1}(Z)) \circ \pi_T \quad \omega_h := \omega \circ (\pi_T)_* = \pi_T^*(\omega). \tag{4.13}$$

The horizontal lift of ω has the properties

$$\omega_h(Y^v) = 0 \quad \text{for} \quad Y \in \mathcal{X}(M) \tag{4.14a}$$

$$\omega_h(Y^h) = (\omega(Y)) \circ \pi_T \quad \text{for} \quad Y \in \mathcal{X}(M), \tag{4.14b}$$

the first of which is equivalent to

$$\omega_h(Z) = 0 \quad \text{for} \quad Z \in \Delta^v, \tag{4.14a$'$}$$

due to proposition 4.1.

Thus there arises a lift $\mathcal{X}(M) \to \mathcal{X}(T(M))$ such that the *lift* of $Y \in \mathcal{X}(M)$ is $\bar{Y} \in \mathcal{X}(T(M))$ with

$$\bar{Y} := Y^v \oplus Y^h. \tag{4.15}$$

Obviously, this decomposition respects definition 4.1 and

$$(\pi_T)_*(\bar{Y}) = (\pi_T)_*(Y^h) = Y. \tag{4.16}$$

The dual *lift* $\omega \mapsto \bar{\omega} \in \Lambda^1(T(M))$ of a 1-form $\omega \in \Lambda^1(M)$ is given by

$$\bar{\omega} = \omega_v \oplus \omega_h. \tag{4.17}$$

As a result of (4.4) and (4.14), we have

$$\bar{\omega}(\bar{Y}) = \omega_v(Y^v) + \omega_h(Y^h) = 2(\omega(Y)) \circ \pi_T. \tag{4.18}$$

At last, let us look on the vertical and/or horizontal lifts from the view point of local bases/frames.

In a case of the tangent bundle $(T(M), \pi_T, M)$ (resp. cotangent bundle $(T^*(M), \pi_T^*, M)$) over a manifold M, any coordinate system $\{x^\mu\}$ on an open set $U_M \subseteq M$ induces natural vector bundle coordinates in the bundle space [5, sec. 1.25] (see also [27, pp. 8, 43]). For

the purpose, we put $e_\mu = \frac{\partial}{\partial x^\mu}$, so that $e^\mu = \mathrm{d}x^\mu$ and we get $(\lambda, \mu, \ldots = 1, \ldots, \dim M$ and $a, b = \dim M + 1, \ldots, 2\dim M)$

$$\{u^I\} = \{x^\mu \circ \pi_T, \mathrm{d}x^\nu\} \quad \text{i.e.} \quad u^\mu = x^\mu \circ \pi_T \quad u^a = \mathrm{d}x^{a-\dim M} \tag{4.19a}$$

on $\pi_T^{-1}(U_M)$ in the tangent bundle case, and

$$\{u^I\} = \Big\{x^\mu \circ \pi_{T^*}, (\cdot)\Big(\frac{\partial}{\partial x^\nu}\Big)\Big\} \quad \text{i.e.} \quad u^\mu = x^\mu \circ \pi_{T^*} \quad u^{\dim M+\nu}\colon \xi \mapsto \xi\Big(\frac{\partial}{\partial x^\nu}\Big) \tag{4.19b}$$

on $\pi_{T^*}^{-1}(U_M) \ni \xi$, in the cotangent bundle case. In connection with the higher order (co)tangent bundles, it is convenient the vector fibre coordinates to be denoted also as $u_1^\mu := \dot{x}^\mu := \mathrm{d}x^\mu$ in $T(M)$ and by $u_\mu^1(\cdot) = (\cdot)\Big(\frac{\partial}{\partial x^\mu}\Big)$ in $T^*(M)$.

Consider the vector bundle coordinates $\{u^\mu = x^\mu \circ \pi_T, u_1^\nu = \mathrm{d}x^\nu\}$ on $\pi_T^{-1}(U_M)$. They induce the frame $\big\{\partial_\mu = \frac{\partial}{\partial u^\mu}, \partial_\nu^1 = \frac{\partial}{\partial u_1^\nu}\big\}$ and the coframe $\{\mathrm{d}u^\mu, \mathrm{d}u_1^\nu\}$ on $\pi_T^{-1}(U_M)$ and $\pi_{T^*}^{-1}(U_M)$, respectively. According to (3.30), they induce the following adapted frame and its dual coframe:

$$(X_\mu, X_\mu^1) = (\partial_\nu, \partial_\nu^1) \cdot \begin{bmatrix} \delta_\mu^\nu & 0 \\ +\Gamma_\mu^\nu & \delta_\mu^\nu \end{bmatrix} = (\partial_\mu + \Gamma_\mu^\nu \partial_\nu^1, \partial_\mu^1) \tag{4.20a}$$

$$\begin{pmatrix} \omega^\mu \\ \omega_1^\mu \end{pmatrix} = \begin{bmatrix} \delta_\nu^\mu & 0 \\ -\Gamma_\nu^\mu & \delta_\nu^\mu \end{bmatrix} \cdot \begin{pmatrix} \mathrm{d}u^\nu \\ \mathrm{d}u_1^\mu \end{pmatrix} = \begin{pmatrix} \mathrm{d}u^\mu \\ \mathrm{d}u_1^\mu - \Gamma_\nu^\mu \mathrm{d}u^\nu \end{pmatrix}, \tag{4.20b}$$

where, as accepted in the (co)tangent bundle case, a fibre index, like a, is replace with a base index, like μ, according to $a \mapsto \mu = a - \dim M$, which leads to identification like $\Gamma_\mu^\nu := \Gamma_\mu^{\dim M+\nu}$.

Consider a vector field $Y = Y^\mu \frac{\partial}{\partial x^\mu} \in \mathcal{X}(M)$ and 1-form $\eta = \eta_\mu \mathrm{d}x^\mu \in \Lambda^1(M)$. According to (4.2) and (4.5), their vertical lifts are

$$Y^v = (Y^\mu \circ \pi_T) X_\mu^1 \in \Delta^v \quad \eta^v = (\eta_\mu \circ \pi_{T^*}) \omega_1^\mu \tag{4.21a}$$

and similarly, due to (4.11), the horizontal lifts of Y and η are

$$Y^h = (Y^\mu \circ \pi_T) X_\mu \in \Delta^h \quad \eta^h = (\eta_\mu \circ \pi_{T^*}) \omega^\mu. \tag{4.21b}$$

4.3. Linear connections on vector bundles

The most valued structures in/on vector bundles are the ones which are compatible/consistent with the linear structure of the fibres of these bundles. Since a distribution $\Delta\colon p \mapsto \Delta_p \subseteq T_p(E)$, $p \in E$, on the bundle space E of a (vector) bundle (E, π, M) cannot be considered as a linear mapping without additional hypotheses, the concept of a linear connection arises from the one of the parallel transport assigned to a connection (see definition 3.2). (For an alternative approach, see [21, p. 42].)

Definition 4.1. A connection on a vector bundle is called *linear* if the assigned to it parallel transport is a linear mapping along every path in the base space, i.e. if the mapping (3.13) is linear for all paths $\gamma\colon [\sigma, \tau] \to M$ in the base.

The restriction on a connection to be linear is quite severe and is described locally by

Theorem 4.1 (cf. [20, sec. 5.2] **).** *Let (E, π, M) be a vector bundle, $\{u^I\}$ be vector bundle coordinates on an open set $U \subseteq E$, and Δ^h be a connection on it described in the frame*

$\{X_I\}$, adapted to $\{u^I\}$, by its 2-index coefficients Γ^a_μ (see (3.27)–(3.29)). The connection Δ^h is linear if and only if, for each $p \in U$,

$$\Gamma^a_\mu(p) = -\Gamma^a_{b\mu}(\pi(p))u^b(p) = -\big((\Gamma^a_{b\mu} \circ \pi) \cdot u^b\big)(p), \tag{4.22}$$

where $\Gamma^a_{b\mu}\colon \pi(U) \to \mathbb{K}$ are some functions on the set $\pi(U) \subseteq M$ and the minus sign before $\Gamma^a_{b\mu}$ in (4.22) is conventional.

Proof. Take a C^1 path $\gamma\colon [\sigma,\tau] \to \pi(U)$ and consider the parallel transport equation (3.39″a), viz.

$$\frac{\mathrm{d}\bar\gamma^a_p(t)}{\mathrm{d}t} = \Gamma^a_\mu(\bar\gamma_p(t))\dot\gamma^\mu(t), \tag{4.23}$$

where $\bar\gamma_p\colon [\sigma,\tau] \to U$ is the horizontal lift of γ through $p \in \pi^{-1}(\gamma(\sigma))$, $\bar\gamma^a := u^a \circ \bar\gamma$, and $\dot\gamma^\mu(t) = \frac{\mathrm{d}(x^\mu \circ \gamma(t))}{\mathrm{d}t} = \frac{\mathrm{d}(u^\mu \circ \bar\gamma(t))}{\mathrm{d}t}$ as $u^\mu = x^\mu \circ \pi$ for some coordinates $\{x^\mu\}$ on $\pi(U)$.

SUFFICIENCY. If (4.22) holds, (4.23) is transformed into

$$\frac{\mathrm{d}\bar\gamma^a_p(t)}{\mathrm{d}t} = -\Gamma^a_{b\mu}(\gamma(t))\bar\gamma^b_p(t)\dot\gamma^\mu(t), \tag{4.24}$$

which is a system of r linear-first order ordinary differential equations for the r functions $\bar\gamma^{n+1}_p, \dots, \bar\gamma^{n+r}_p$. According to the general theorems of existence and uniqueness of the solutions of such systems [29], it has a unique solution

$$\bar\gamma^a_p(t) = Y^a_b(t)p^b \tag{4.25}$$

satisfying the initial condition $\bar\gamma^a_p(\sigma) = u^a(p) =: p^a$, where $Y = [Y^a_b]$ is the fundamental solution of (4.24), i.e.

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t} = -[\Gamma^a_{b\mu}(\gamma(t))\dot\gamma^\mu(t)]^{n+r}_{a,b=n+1} \cdot Y(t) \quad Y(\sigma) = \mathbb{1}_{r\times r} = [\delta^a_b]. \tag{4.26}$$

The linearity of (3.13) in p follows from (4.25) for $t = \tau$.

NECESSITY. Suppose (3.13) is linear in p for all paths $\gamma\colon [\sigma,\tau] \to \pi(U)$. Then $\bar\gamma_p(t) := \mathsf{P}^{\gamma|[\sigma,t]}(p)$ is the horizontal lift of $\gamma|[\sigma,t]$ through p and (cf. (4.25)) $\bar\gamma^a_p(t) = A^a_b(\gamma(t))p^b$ for some C^1 functions $A^a_b\colon \pi(U) \to \mathbb{K}$. The substitution of this equation in (4.23) results into

$$\frac{\partial A^a_b(x)}{\partial x^\mu}\Big|_{x=\gamma(t)=\pi(\bar\gamma_p(t))} \cdot \dot\gamma^\mu p^b = \Gamma^a_\mu(\bar\gamma_p(t))\dot\gamma^\mu(t).$$

Since $\gamma\colon [\sigma,\tau] \to M$, we get equation (4.22) from here, for $t = \sigma$, with $\Gamma^a_{b\mu}(x) = -\frac{\partial A^a_b(x)}{\partial x^\mu}$ for $x \in \pi(U)$. □

The functions $\Gamma^a_{b\mu}\colon \pi(U) \to \mathbb{K}$ will be referred as the *(local) 3-index coefficients* of the linear connection Δ^h in the adapted frame $\{X_I\}$. If there is no risk to confuse them with the 2-index coefficients $\Gamma^a_\mu\colon U \to \mathbb{K}$, they will be called simply coefficients of Δ^h. Note, the 2-index coefficients of a linear connections are defined on (a subset of) the bundle space E, while the 3-index ones are define on (a subset of) the base space M. The equation (4.24) is simply the *parallel transport equation* for the linear connection considered.

Example 4.1. Since u^a is replaced by $u^\mu_1 = \mathrm{d}x^\mu$ in the tangent bundle case (see Subsect. 4.2), the linear connections in $(T(M), \pi_T, M)$ have 2-index coefficients of the form

$$\Gamma^\nu_\mu = -(\Gamma^\nu_{\lambda\mu} \circ \pi_T) \cdot u^\lambda_1 = -(\Gamma^\nu_{\lambda\mu} \circ \pi_T) \cdot \mathrm{d}x^\lambda \tag{4.27}$$

and, consequently, they can be regarded as 1-forms on M.

Consider a linear connection Δ^h on a vector bundle (E,π,M). Let Γ^a_μ and $\Gamma^a_{b\mu}$ be its 2- and 3-index coefficients, respectively, in a frame $\{X_I\}$ adapted to vector bundle coordinates $\{u^I\}$.

Corollary 4.1. *The 3-index coefficients $\Gamma^a_{b\mu}$ of a linear connection Δ^h uniquely define the fibre coefficients of Δ^h in $\{X_I\}$ by*

$$ {}^\circ\Gamma^a_{b\mu} = \Gamma^a_{b\mu}\circ\pi = \pi^*(\Gamma^a_{b\mu}), \tag{4.28}$$

that is the fibre coefficients of a linear connection are equal to the 3-index ones lifted by the projection π.

Proof. Since (3.28a) and (4.22) imply

$$[X_\mu, X_b]_- = (\Gamma^a_{b\mu}\circ\pi)X_a, \tag{4.29}$$

the equation (4.28) follows from (3.22a) and (3.23a) or (3.37b) and (4.22). □

As the vector bundle coordinates $\{u^I\}$ are, by definition, linear on the fibres of the bundle, the general change of such coordinates is

$$\{u^\mu, u^a\} \mapsto \{\tilde{u}^\mu = \tilde{x}^\mu\circ\pi, \tilde{u}^a = (B^a_b\circ\pi)\cdot u^b\}, \tag{4.30}$$

with $B = [B^a_b]$ being a non-degenerate matrix-valued function on $\pi(U)$. The change (4.30) entails the following transformation of the corresponding adapted frames

$$\{X_\mu, X_a\} \mapsto \{\tilde{X}_\mu = (B^\nu_\mu\circ\pi)\cdot X_\nu, \tilde{X}_a = (B^b_a\circ\pi)\cdot X_b\}, \tag{4.31}$$

where $[B^\nu_\mu] = \left[\frac{\partial x^\nu}{\partial\tilde{x}^\mu}\right]$ is a non-degenerate matrix-valued function on the intersection of the domains of $\{x^\mu\}$ and $\{\tilde{x}^\mu\}$. (In (4.31) we have used that $\frac{\partial u^\nu}{\partial\tilde{u}^\mu}\big|_p = \frac{\partial(x^\nu\circ\pi)}{\partial(\tilde{x}^\mu\circ\pi)}\big|_p = \frac{\partial x^\nu}{\partial\tilde{x}^\mu}\big|_{\pi(p)}$.)

Proposition 4.2. *The change* (4.30) *implies the following transformations of the 3-index coefficients of the linear connection:*

$$\Gamma^a_{b\mu} \mapsto \tilde{\Gamma}^a_{b\mu} = B^\nu_\mu\Big(B^a_d\Gamma^d_{c\nu} - \frac{\partial B^a_c}{\partial x^\nu}\Big)(B^{-1})^c_b. \tag{4.32}$$

Proof. Apply (4.31), (3.32) and (4.22). Alternatively, the same transformation law follows also from equations (3.24a) and (4.28). □

If we introduce the matrix-valued functions $\Gamma_\mu := [\Gamma^a_{b\mu}]$ and $\tilde{\Gamma}_\mu := [\tilde{\Gamma}^a_{b\mu}]$ on M, we can rewrite (4.32) as

$$\begin{aligned}\Gamma_\mu \mapsto \tilde{\Gamma}_\mu &= B^\nu_\mu\left(B\cdot\Gamma_\nu - \frac{\partial B}{\partial x^\nu}\right)\cdot B^{-1}\\ &= B^\nu_\mu B\cdot\left(\Gamma_\nu\cdot B^{-1} + \frac{\partial B^{-1}}{\partial x^\nu}\right).\end{aligned} \tag{4.32$'$}$$

This relation correspond to (3.25) with $[A^a_b] = B^{-1}\circ\pi$ (see also (4.28)) as the frame $\{e_a\colon M\to E\}$, relative to which the vector fibre coordinates $\{u^a\}$ are defined ($E\ni p\mapsto u^a(p)$ with $p = u^a(p)e_a(\pi(p))$), transforms via the matrix inverse to $B\circ\pi$.

Let E be a C^2 manifold and Δ^h a C^1 connection on (E,π,M). Substituting (4.22) into (3.37a), we get the fibre components of the curvature of a linear connection as

$$R^a_{\mu\nu} = -(R^a_{b\mu\nu}\circ\pi)\cdot u^b \tag{4.33}$$

where

$$R^a_{b\mu\nu} := \frac{\partial}{\partial x^\mu}(\Gamma^a_{b\nu}) - \frac{\partial}{\partial x^\nu}(\Gamma^a_{b\mu}) - \Gamma^c_{b\mu}\Gamma^a_{c\nu} + \Gamma^c_{b\nu}\Gamma^a_{c\mu}, \tag{4.34}$$

or in a matrix form

$$R_{\mu\nu} := [R^a_{b\mu\nu}] = \frac{\partial \Gamma_\nu}{\partial x^\mu} - \frac{\partial \Gamma_\mu}{\partial x^\nu} - \Gamma_\nu \cdot \Gamma_\mu + \Gamma_\mu \cdot \Gamma_\nu, \tag{4.34$'$}$$

are the *components of the curvature operator* (see below (4.52)). As a result of (3.22b) and (4.33), the transformation (4.30) entails the change

$$R^a_{b\mu\nu} \mapsto \tilde{R}^a_{b\mu\nu} = B^\lambda_\mu B^\varrho_\nu (B^{-1})^a_c B^d_b R^c_{d\lambda\varrho}, \tag{4.35}$$

or in a matrix form

$$R_{\mu\nu} \mapsto \tilde{R}_{\mu\nu} = B^\lambda_\mu B^\varrho_\nu B^{-1} \cdot R_{\lambda\varrho} \cdot B, \tag{4.35$'$}$$

which corresponds to (3.24b) with $A = B^{-1} \circ \pi$ (see also (4.33)).

4.4. Covariant derivatives in vector bundles

A possibility for introduction of differentiation in vector bundles, endowed with connection, comes from the vector space structure of their fibres. This operation can be defined in many independent ways, leading to identical results. In one of them is involved the parallel transport induced by the connection: the idea is the values of sections to be parallel transported (along paths in the base) into a single fibre (over the paths), where one can work with the 'transported' sections as with functions with values in a vector space. Other method uses the existence of natural vertical lifts of sections of the bundle and horizontal lifts of the vector fields on the base space; since the both lifts are vector fields on the bundle space, their commutator (or Lie derivative relative to each other) is well defined and can be used as a prototype of some sort of differentiation. We shall realize below the second method mentioned, which seems is first introduced in a rudimentary form in [20, p. 31]. [13] [14] The first way, as well as the axiomatic approach, for introduction of covariant derivatives will be obtained as theorems in what follows.

Let (E, π, M) be a vector bundle on which a *linear* connection Δ^h is defined. Suppose $\{E_a\}$ is a frame in E to which vector fibre coordinates $\{u^a\}$ are associated and $\{u^I\}$ be the corresponding vector bundle coordinates. The frame adapted to $\{u^I\}$ will be denoted by $\{X_I\}$ and $\{\omega^I\}$ will be its dual coframe, both defined by (3.28) through the (2-index) coefficients Γ^a_μ of Δ^h.

Let $\hat{Z} = \hat{Z}^a X_a \in \Delta^v$ and $\bar{Z} = \bar{Z}^\mu X_\mu \in \Delta^h$ be respectively vertical and horizontal vector fields on E. Define a mapping $\hat{\nabla}\colon \Delta^v \oplus \Delta^h = T(E) \to \mathcal{X}(E)$ such that [15]

$$\hat{\nabla}\colon (\hat{Z}, \bar{Z}) \mapsto \hat{\nabla}_{\bar{Z}}(\hat{Z}) := \Pi(\mathcal{L}_{\bar{Z}}\hat{Z}) \in \mathcal{X}(E), \tag{4.36}$$

where the (1,1) tensor field

$$\Pi := \sum_a X_a \otimes \omega^a \tag{4.37}$$

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[13] In [20, p. 31] is proved that, for $F = \frac{\partial}{\partial x^\mu}$ and in our notation, the a-th component of the right hand sides of (4.44) and of (4.45) coincide in a frame $\{E_a\}$ in E.

[14] An equivalent alternative approach is realized in [23, sections 2.49–2.52].

[15] The idea of the construction (4.36) is to drag the vertical vector field $\hat{Z}$ along the horizontal one $\bar{Z}$, which will give a vector field in $\mathcal{X}(E)$, and then to project the result onto the *vertical* distribution Δ^v by means of the invariant projection operator $\Pi = X_a \otimes \omega^a \colon \mathcal{X}(E) \to \mathcal{X}(E)$. Evidently $\Pi^2 = \Pi \circ \Pi = \Pi$ and Π is the unit (identity) tensor in the tensor product of vector fields and 1-forms on E.

is considered as a operator on the set of vector fields on E. Since (see (2.1b) and (2.7))

$$\mathcal{L}_{\bar{Z}}\hat{Z} = \bar{Z}(\hat{Z}^a)X_a + \bar{Z}^\mu \hat{Z}^a [X_\mu, X_a]_-$$

and $\omega^a(X_\mu) = \delta^a_\mu = 0$, form (3.36), (3.37b) and (4.36), we obtain

$$\hat{\nabla}_{\bar{Z}}\hat{Z} = \bar{Z}^\mu \{X_\mu(\hat{Z}^a) - \hat{Z}^b \partial_b(\Gamma^a_\mu)\} X_a, \tag{4.38}$$

from where one can prove, via direct calculation, the independence of $\hat{\nabla}_{\bar{Z}}\hat{Z}$ of the particular (co)frame used. For any particular point $p \in E$, the value of the vector field (4.38) is a vertical vector, $(\hat{\nabla}_{\bar{Z}}\hat{Z})|_p \in \Delta^v_p$, but generally $\hat{\nabla}_{\bar{Z}}\hat{Z}$ is not a vertical vector field. The reason is that a vertical vector field on E is a mapping $V\colon p \mapsto V_p \in \Delta^v_p := T_p(\pi^{-1}(\pi(p)) := T_{\imath(p)}(\pi^{-1}(\pi(p)) = (\pi_*|_p)^{-1}(0_{\pi(p)})$, with $\imath\colon \pi^{-1}(p) \to E$ being the inclusion mapping and $0_{\pi(p)} \in T_{\pi(p)}(M)$ being the zero vector, due to which V_p, and hence its components, must depend only on $\pi(p) \in M$. Therefore, we have

$$\hat{\nabla}_{\bar{Z}}\hat{Z} \in \Delta^v \iff \partial_b(\Gamma^a_\mu) = -\Gamma^a_{b\mu} \circ \pi \iff \Gamma^a_\mu = -(\Gamma^a_{b\mu} \circ \pi) \cdot u^b + G^a_\mu \circ \pi, \tag{4.39}$$

for some functions $\Gamma^a_{b\mu}, G^a_\mu\colon M \to \mathbb{K}$. Thus $\hat{\nabla}_{\bar{Z}}\hat{Z}$ is a vertical vector field if and only if the 2-index coefficients Γ^a_μ in $\{X_I\}$ of the connection Δ^h are of the form

$$\Gamma^a_\mu = -(\Gamma^a_{b\mu} \circ \pi) \cdot u^b + G^a_\mu \circ \pi. \tag{4.40}$$

This equality selects the set of *affine connections* among all connections (see Subsect 4.5 below); [16] in particular, of this type are the linear connections for which $G^a_\mu = 0$ and $\Gamma^a_{b\mu}$ are their 3-index coefficients (see (4.22)). For connections with 2-index coefficients (4.40), equation (4.38) reduces to

$$\hat{\nabla}_{\bar{Z}}\hat{Z} = \bar{Z}^\mu \{X_\mu(\hat{Z}^a) + \hat{Z}^b(\Gamma^a_{b\mu} \circ \pi)\} X_a \in \Delta^v. \tag{4.41}$$

Now the idea of introduction of a covariant derivative of a section $Y \in \mathrm{Sec}(E, \pi, M)$ along a vector field $F \in \mathcal{X}(M)$ is to 'lower the operator $\hat{\nabla}$ from $T(E)$ to $T(M)$.

Definition 4.2. A *covariant derivative* or *covariant derivative operator*, associated to a linear (or affine) connection Δ^h on a vector bundle (E, π, M), is a mapping

$$\begin{aligned} \nabla\colon & \mathcal{X}(M) \times \mathrm{Sec}^1(E, \pi, M) \to \mathrm{Sec}^0(E, \pi, M) \\ \nabla\colon & (F, Y) \mapsto \nabla_F Y \end{aligned} \tag{4.42}$$

such that, for $F \in \mathcal{X}(M)$ and $Y \in \mathrm{Sec}^1(E, \pi, M)$, $\nabla_F Y$ is the unique section of (E, π, M) whose vertical lift is $\hat{\nabla}_{F^h} Y^v$, with $\hat{\nabla}$ defined by (4.36) (or (4.41)), viz.

$$(\nabla_F Y)^v := \hat{\nabla}_{F^h} Y^v \tag{4.43}$$

or

$$(\nabla_F Y) = v^{-1} \circ \hat{\nabla}_{(\pi_*|_{\Delta_h})^{-1}(F)}(v(Y)), \tag{4.44}$$

where $F^h \in \Delta^h$ and $Y^v \in \Delta^v$ are respectively the horizontal and vertical lifts of F and Y.

Remark 4.1. Definition 4.2 and the rest of this subsection are valid also for affine connections for which (4.40) holds, not only for the linear ones. For some details, see Subsect. 4.5.

[16] Usually the affine connections are defined on affine bundles [2, 21].

Proposition 4.3. *Let $\{E_a\}$ be a frame in E and $\{x^\mu\}$ local coordinates on M. If $Y = Y^a E_a \in \operatorname{Sec}^1(E,\pi,M)$ and $F = F^\mu \frac{\partial}{\partial x^\mu} \in \mathcal{X}(M)$, then we have the explicit local expression*

$$\nabla_F Y = F^\mu \Big(\frac{\partial Y^a}{\partial x^\mu} + \Gamma^a_{b\mu} Y^b \Big) E_a. \tag{4.45}$$

Proof. Apply (4.43), (4.10), (4.11), (4.41), and (4.2). □

Proposition 4.4. *Let Δ^h be a linear connection on (E,π,M) and P be the generated by it parallel transport. Let $x \in M$, $\gamma\colon [\sigma,\tau] \to M$, $\gamma(t_0) = x$ for some $t_0 \in [\sigma,\tau]$, and $\dot\gamma(t_0) = F_x$, i.e. γ to be the integral path of $F \in \mathfrak{X}(M)$ through x. Then*

$$(\nabla_F Y)|_x = \lim_{s\to t_0} \frac{P^\gamma_{s\to t_0}(Y_{\gamma(s)}) - Y_{\gamma(t_0)}}{s - t_0} = \lim_{\varepsilon\to 0} \frac{P^\gamma_{t_0+\varepsilon\to t_0}(Y_{\gamma(t_0+\varepsilon)}) - Y_{\gamma(t_0)}}{\varepsilon}, \tag{4.46}$$

where $Y \in \operatorname{Sec}^1(E,\pi,M)$ and

$$P^\gamma_{s\to t} := \begin{cases} \mathsf{P}^{\gamma|[s,t]} & \text{for } s \le t \\ \big(\mathsf{P}^{\gamma|[t,s]}\big)^{-1} & \text{for } s \ge t. \end{cases} \tag{4.47}$$

Proof. Use definition 3.2 and apply the parallel transport equation (4.24) with initial value $\bar\gamma_{Y_{\gamma(s)}}(s) = Y_{\gamma(s)}$ at the point $t = s \in [\sigma,\tau]$. □

By proposition 4.4, the equation (4.46) can be used as an equivalent definition of a covariant derivative associated with a linear connection.

Proposition 4.5. *Let $F, G \in \mathcal{X}(M)$, $Y, Z \in \operatorname{Sec}^1(E,\pi,M)$, and $f\colon M \to \mathbb{K}$ be a C^1 function. Then:*

$$\nabla_{F+G} Y = \nabla_F Y + \nabla_G Y \tag{4.48a}$$
$$\nabla_{fF} Y = f \nabla_F Y \tag{4.48b}$$
$$\nabla_F (Y + Z) = \nabla_F Y + \nabla_F Z \tag{4.48c}$$
$$\nabla_F (fY) = F(f) \cdot Y + f \cdot \nabla_F Y. \tag{4.48d}$$

Proof. Apply (4.45). □

Proposition 4.6. *If a mapping* (4.42) *satisfies* (4.48)*, there exists a unique linear connection Δ^h, the assigned to which covariant derivative is exactly ∇.*

Proof. Define local functions $\Gamma^a_{b\mu}$ on M, called *components* of ∇, by the decomposition

$$\nabla_{\frac{\partial}{\partial x^\mu}} E_b =: \Gamma^a_{b\mu} E_a. \tag{4.49}$$

A simple verification proves that they transform according to (4.32) and hence the quantities (4.22) transform by (3.32). Proposition 3.3 ensures the existence of a unique linear connection whose 2-index (3-index) coefficients are Γ^a_μ ($\Gamma^a_{b\mu}$). Thus the covariant derivative of $Y \in \operatorname{Sec}(E,\pi,M)$ relative to $F \in \mathcal{X}(M)$ is given by the r.h.s. of (4.45). On another hand, (4.48) entail (4.45), with $\Gamma^a_{b\mu}$ defined by (4.49), so that ∇ is exactly the covariant derivative operator assigned to the connection with 3-index coefficients $\Gamma^a_{b\mu}$. □

Consequently, equations (4.48) and (4.49) provide a third equivalent definition of a covariant derivative (covariant derivative operator). Moreover, since proposition 4.6 establishes a bijective correspondence between linear connections and operators (4.42) satisfying (4.48), quite often such operators are called linear connections. [17] As it is clear from the proof of proposition 4.6, the bijection between linear connection and covariant derivative operators is locally given by the coincidence of their (3-index) coefficients and components, respectively.

[17] See also [23, sections 2.15 and 2.52].

Exercise 4.1. A C^1 section $\omega = \omega_a E^a$ of the bundle dual to (E, π, M) can be differentiated covariantly similarly to the sections of (E, π, M). Show that the corresponding operator, say ∇^*, can equivalently be defined by (the 'Leibnitz rule')

$$(\nabla^*_F \omega)(Y) = F(\omega(Y)) - \omega(\nabla_F Y) \tag{4.50}$$

and locally is valid the equation

$$\nabla^*_F \omega = F^\mu \Big(\frac{\partial \omega_a}{\partial x^\mu} - \Gamma^b_{a\mu} \omega_b \Big) E^a. \tag{4.51}$$

Equipped with the covariant derivative ∇ assigned to a C^1 linear connection Δ^h, we define the *curvature operator* of Δ^h (or ∇) by

$$\begin{aligned} &R\colon \mathcal{X}(M) \times \mathcal{X}(M) \to \operatorname{End}(\operatorname{Sec}(E, \pi, M)) \\ &R\colon (F, G) \mapsto R(F, G) := \nabla_F \circ \nabla_G - \nabla_G \circ \nabla_F - \nabla_{[F,G]_-}, \end{aligned} \tag{4.52}$$

with $\operatorname{End}(\dots)$ denoting the set of endomorphisms of $(\dots)$.

Exercise 4.2. Prove that locally

$$(R(F, G))(Y) = (R^a_{b\mu\nu} Y^b F^\mu G^\nu) E_a, \tag{4.53}$$

where the functions $R^a_{b\mu\nu}\colon M \to \mathbb{K}$, called the *components* of the curvature operator R in the pair of frames $\big(\{\frac{\partial}{\partial x^\mu}\}, \{E_a\}\big)$, are defined by

$$R\Big(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\Big)(E_b) =: R^a_{b\mu\nu} E_a \tag{4.54}$$

and are explicitly expressed through the coefficients of ∇(= 3-index coefficients of Δ^h) via (4.34).

A linear connection or covariant derivative operator is called *flat* or *curvature free* if

$$R = 0 \qquad (\iff R^a_{b\mu\nu} = 0). \tag{4.55}$$

Obviously, the flatness of Δ^h or ∇ is a necessary and sufficient condition for the (local) integrability of the horizontal distribution $\Delta^h\colon p \mapsto \Delta^h_p \subseteq T_p(E)$, $p \in E$ (see (3.23b) and (4.33)).

Theorem 4.2. *Let Y be a C^1 section of a vector bundle (E, π, M) endowed with a linear connection Δ^h. The following three conditions are equivalent:*

(i) *Y is covariantly constant, viz., if $F \in \mathcal{X}(M)$, then*

$$\nabla_F Y = 0. \tag{4.56}$$

(ii) *Y is a solution of Δ^h, i.e.*

$$\operatorname{Im} Y_* \subset \Delta^h \quad (\iff Y_*|_x(T_x(M)) \subseteq \Delta^h_{Y_x} \text{ for } x \in M). \tag{4.57}$$

(iii) *Y is parallelly transported along any path $\gamma\colon [\sigma, \tau] \to M$,*

$$\mathsf{P}^\gamma(Y_{\gamma(\sigma)}) = Y_{\gamma(\tau)}. \tag{4.58}$$

Proof. Since $Y = u^a(Y)e_a$, $\pi \circ Y = \mathsf{id}_M$, and $\omega^a = \mathrm{d}u^a - \Gamma^a_\mu \mathrm{d}u^\mu$, we have for $x \in M$:

$$\omega^a \circ Y\Big(\frac{\partial}{\partial x^\mu}\Big) = \omega^a\Big(\frac{\partial(u^\nu \circ Y)}{\partial x^\mu}\Big|_x \frac{\partial}{\partial u^\nu}\Big|_{Y_x} + \frac{\partial(u^a \circ Y)}{\partial x^\mu}\Big|_x \frac{\partial}{\partial u^a}\Big|_{Y_x}\Big)$$
$$= -\frac{\partial(x^\nu \circ \pi \circ Y)}{\partial x^\mu}\Big|_x \Gamma^a_\nu(Y_x) + \frac{\partial Y^a}{\partial x^\mu}\Big|_x = \Big(\frac{\partial Y^a}{\partial x^\mu} - \Gamma^a_\mu \circ Y\Big)(x).$$

The equivalence of (i) and (ii) follows from here, (4.22), (4.45), and that Δ^h annihilates the 1-forms ω^a, $\omega^a(Z) \iff Z \in \Delta^h$.

If we rewrite the parallel transport equation (4.24) as (see (4.45))

$$(\nabla_{\dot\gamma(t)}\bar\gamma)|_{\gamma(t)} = 0, \tag{4.59}$$

the equivalence of (i) and (iii) follows from definition 3.2 of a parallel transport and the arbitrariness of γ in (4.59). □

Exercise 4.3. Formulate and prove a theorem dual to theorem 4.2; e.g. a section $\varphi = \varphi_a u^a$ of the bundle dual to (E, π, M) is a first integral of Δ^h, i.e. $\operatorname{Ker}\varphi_* \supseteq \Delta^h$ ($\iff \varphi_*|_p(\Delta^h_p) = 0_{\varphi(p)} \in T_{\varphi(p)}(\mathbb{K})$ for $p \in E$), if and only if

$$\nabla^*\varphi = 0. \tag{4.60}$$

Proposition 4.7 (cf. [20, p. 32] **).** *Let a linear connection Δ^h on a vector bundle be given and $\Gamma^a_{b\mu}$ be its (3-index) coefficients. The following conditions are (locally) equivalent:*

(a) *Δ^h is integrable.*

(b) *Δ^h is flat.*

(c) *There exists a solution of the system of partial differential equations*

$$\frac{\partial U^a}{\partial x^\mu} + \Gamma^a_{b\mu}U^b = 0 \tag{4.61}$$

relative to U^a and the solution of (4.61) satisfying $U^a|_{x=x_0} = U^a_0$ is $U^a = B^a_b U^b_0$, where $B = [B^a_b]$ is the fundamental solution of (4.61), viz.

$$\frac{\partial B^a_b}{\partial x^\mu} + \Gamma^a_{c\mu}B^c_b = 0 \qquad B^a_b|_{x=x_0} = \delta^a_b. \tag{4.62}$$

(d) *There is an integrating matrix B^{-1} for the 1-forms ω^a, that is $(B^{-1}\circ\pi)^a_b\omega^b = \mathrm{d}f^a$, where the functions $f^a\colon E \to \mathbb{K}$ are first integrals of Δ^h, i.e.* $\operatorname{Ker} f^a \supset \Delta^h$.

(e) *The coefficients of Δ^h have the form*

$$\Gamma_\mu := [\Gamma^a_{b\mu}] = B \cdot \frac{\partial B^{-1}}{\partial x^\mu} = -\frac{\partial B}{\partial x^\mu}\cdot B^{-1} \tag{4.63}$$

for some matrix-valued function B on M.

Proof. (a) $\iff$ (b): See (4.55) and the comment after it.

(c) $\iff$ (e): The matrix form of the equation in (4.62), i.e.

$$\frac{\partial B}{\partial x^\mu} + \Gamma_\mu \cdot B = 0, \tag{4.62$'$}$$

is tantamount to (4.63).

(b) $\iff$ (c): The flatness of Δ^h, i.e. $R_{\mu\nu} = 0$ (see (4.34′)), is the integrability condition for (4.62′) as an equation relative to B – see [28, lemma 2.1].

(c) $\Longleftrightarrow$ (d): Since (3.36) and the first equality in (2.1c) entail

$$\mathcal{L}_{X_\mu}(\varphi_a\omega^a) = -\varphi_a R^a_{\mu\nu}\omega^\nu + \{X_\mu(\varphi_a) - {}^\circ\Gamma^b_{a\mu}\varphi_b\}\omega^a, \tag{4.64}$$

we have (see also (4.28)) for a flat linear connection:

$$\begin{aligned}\mathcal{L}_{X_\mu}((B^{-1})^a_b\omega^b) &= \Big\{\Big(\frac{\partial B^{-1}}{\partial x^\mu} - B^{-1}\cdot\Gamma_\mu\Big)\circ\pi\Big\}^a_b\,\omega^b\\ &= \Big\{\Big[-B^{-1}\cdot\Big(\frac{\partial B}{\partial x^\mu} + \Gamma_\mu\cdot B\Big)B^{-1}\Big]\circ\pi\Big\}^a_b\,\omega^b.\end{aligned}$$

Thus (4.62), which entails (c), is equivalent to $\mathcal{L}_{X_\mu}((B^{-1}\circ\pi)^a_b\,\omega^b) = 0$, which is equivalent to $\mathrm{d}((B^{-1}\circ\pi)^a_b\,\omega^b) = 0$, due to $\omega^a(X_\mu) = \delta^a_\mu = 0$ and the second equality in (2.1a) (applied, e.g., for $Y = X_\nu$). Now the Poincaré's lemma (see [4, sec. 6.3] or [30, pp. 21, 106]) tells us that locally (on a contractible region in E) there are functions f^a on E such that the last equality is tantamount to $\mathrm{d}f^a = (B^{-1}\circ\pi)^a_b\,\omega^b$.

It remains to be proved that $f^a\colon E\to\mathbb{K}$ are first integrals of Δ^h, i.e. $\operatorname{Ker} f^a \supset \Delta^h$ which means $(f^a)_*|_p(\Delta^h_p) = 0$, $p\in E$, or $(f^a)_*|_p(X_\mu) = 0$ as Δ^h is spanned by $\{X_\mu\}$. Using the global chart $(\mathbb{K}, \mathsf{id}_{\mathbb{K}})$ on $\mathbb{K}$, which induces the one-vector frame $\{\frac{\partial}{\partial r}\}$ for $r\in\mathbb{K}$ on $\mathbb{K}$, we have (see (3.28))

$$(f^a)_*|_p(X_\mu) = (f^a)_*|_p\Big(\frac{\partial}{\partial u^\mu} + \Gamma^b_\mu\frac{\partial}{\partial u^b}\Big)\Big|_p = \Big(\frac{\partial f^a}{\partial u^\mu}\Big|_p + \Gamma^b_\mu(p)\frac{\partial f^a}{\partial u^b}\Big|_p\Big)\frac{\mathrm{d}}{\mathrm{d}r}\Big|_{f^a(p)} \equiv 0$$

as $\mathrm{d}f^a = (B^{-1}\circ\pi)^a_b\,\omega^a = (B^{-1}\circ\pi)^a_b(\mathrm{d}u^a - \Gamma^b_\mu\circ\mathrm{d}u^\mu) \equiv \frac{\partial f^a}{\partial u^b}\mathrm{d}u^b + \frac{\partial f^a}{\partial u^\mu}\mathrm{d}u^\mu = \mathrm{d}f^a$. □

4.5. Affine connections

In Subsect. 4.4, we met a class of connections on a vector bundle whose local 2-index coefficients have the form (see (4.40))

$$\Gamma^a_\mu = -(\Gamma^a_{b\mu}\circ\pi)\cdot u^b + G^a_\mu\circ\pi. \tag{4.65}$$

in the frame $\{X_I\}$ adapted to vector bundle coordinates $\{u^I\}$. From $\partial_b\Gamma^a_\mu = -\Gamma^a_{b\mu}$ and (3.32), one derives that the functions $\Gamma^a_{b\mu}$ in (4.65) transform according to (4.32), viz.

$$\Gamma^a_{b\mu}\mapsto\tilde\Gamma^a_{b\mu} = B^\nu_\mu\Big(B^a_d\Gamma^d_{c\nu} - \frac{\partial B^a_c}{\partial x^\nu}\Big)(B^{-1})^c_b. \tag{4.66}$$

when the vector bundle coordinates or adapted frames undergo the change (4.30) or (4.31), respectively. Thus, combining (3.32), (4.66) and (4.65), we see that (4.30) or (4.31) implies the transition

$$G^a_\mu\mapsto\tilde G^a_\mu = B^a_b G^b_\nu B^\nu_\mu. \tag{4.67}$$

Consequently, the functions $\Gamma^a_{b\mu}$ in (4.65) are 3-index coefficients of a linear connection, while G^a_μ in it are the components of a linear mapping $G\colon\mathcal{X}(M)\to\operatorname{End}(\operatorname{Sec}((E,\pi,M)^*))$ such that $G\colon F\mapsto G(F)\colon\omega\mapsto(G(F))(\omega)$, for $F\in\mathcal{X}(M)$ and a section ω of the bundle $(E,\pi,M)^*$ dual to (E,π,M), and $\big(G(\frac{\partial}{\partial x^\mu})\big)(E^a) = G^a_\mu$. The invariant description of the connections with local 2-index coefficients of the type (4.65) is as follows.

Definition 4.3. A connection on a vector bundle is termed *affine connection* if the assigned to it parallel transport P is an affine mapping along all paths $\gamma\colon[\sigma,\tau]\to M$ in the base space, i.e.

$$\mathsf{P}^\gamma(\rho X) = \rho\mathsf{P}^\gamma(X) + (1-\rho)\mathsf{P}^\gamma(\mathbf{0}) \tag{4.68a}$$

$$\mathsf{P}^\gamma(X+Y) = \mathsf{P}^\gamma(X) + \mathsf{P}^\gamma(Y) - \mathsf{P}^\gamma(\mathbf{0}), \tag{4.68b}$$

where $\rho\in\mathbb{K}$, $X,Y\in\pi^{-1}(\gamma(\sigma))$, and $\mathbf{0}$ is the zero vector in the fibre $\pi^{-1}(\gamma(\sigma))$, which is a $\mathbb{K}$-vector space.

Theorem 4.3. *Let (E,π,M) be a vector bundle, $\{u^I\}$ be vector bundle coordinates over an open set $U \subseteq E$, and Δ^h be a connection on it with 2-index coefficients Γ^a_μ in the frame $\{X_I\}$ adapted to $\{u^I\}$. The connection Δ^h is an affine connection if and only if equation* (4.65) *holds for some functions $\Gamma^a_{b\mu}, G^a_\mu \colon \pi(U) \to \mathbb{K}$.*

Proof (cf. the proof of theorem 4.1). Take a C^1 path $\gamma\colon [\sigma,\tau] \to \pi(U)$ and consider the parallel transport equation (3.39″a), viz.

$$\frac{\mathrm{d}\bar\gamma^a_p(t)}{\mathrm{d}t} = \Gamma^a_\mu(\bar\gamma_p(t))\dot\gamma^\mu(t), \tag{4.69}$$

where $\bar\gamma_p\colon [\sigma,\tau] \to U$ is the horizontal lift of γ through $p \in \pi^{-1}(\gamma(\sigma))$, $\bar\gamma^a := u^a \circ \bar\gamma$, and $\dot\gamma^\mu(t) = \frac{\mathrm{d}(x^\mu\circ\gamma(t))}{\mathrm{d}t} = \frac{\mathrm{d}(u^\mu\circ\bar\gamma(t))}{\mathrm{d}t}$ as $u^\mu = x^\mu \circ \pi$ for some coordinates $\{x^\mu\}$ on $\pi(U)$.

SUFFICIENCY. If (4.65) holds, (4.69) is transformed into

$$\frac{\mathrm{d}\bar\gamma^a_p(t)}{\mathrm{d}t} = -\Gamma^a_{b\mu}(\gamma(t))\bar\gamma^b_p(t)\dot\gamma^\mu(t) + G^a_\mu(\gamma(t))\dot\gamma^\mu(t), \tag{4.70}$$

which is a system of r linear inhomogeneous first order ordinary differential equations for the r functions $\bar\gamma^{n+1}_p,\dots,\bar\gamma^{n+r}_p$. According to the general theorems of existence and uniqueness of the solutions of such systems [29], it has a unique solution

$$\bar\gamma^a_p(t) = Y^a_b(t)p^b + y^a(t) \tag{4.71}$$

satisfying the initial condition $\bar\gamma^a_p(\sigma) = u^a(p) =: p^a$, where $Y = [Y^a_b]$ is the fundamental solution of (4.24) (see (4.26)) and $y^a(t)$ is the solution of (4.70) with $y^a(t)$ for $\bar\gamma^a_p(t)$ satisfying the initial condition $y^a(\sigma) = 0$. The affinity of (3.13) in p, i.e. (4.68), follows from (4.71) for $t = \tau$.

NECESSITY. Suppose (3.13) is affine in p for all paths $\gamma\colon [\sigma,\tau] \to \pi(U)$. Then $\bar\gamma_p(t) := \mathsf{P}^{\gamma|[\sigma,t]}(p)$ is the horizontal lift of $\gamma|[\sigma,t]$ through p and (cf. (4.71)) $\bar\gamma^a_p(t) = A^a_b(\gamma(t))p^b + A^a(\gamma(t))$ for some C^1 functions $A^a_b, A^a\colon \pi(U) \to \mathbb{K}$. The substitution of this equation in (4.69) results into

$$\frac{\partial A^a_b(x)}{\partial x^\mu}\Big|_{x=\gamma(t)=\pi(\bar\gamma_p(t))} \cdot \dot\gamma^\mu p^b + \frac{\partial A^a(x)}{\partial x^\mu}\Big|_{x=\gamma(t)=\pi(\bar\gamma_p(t))} \cdot \dot\gamma^\mu(t). = \Gamma^a_\mu(\bar\gamma_p(t))\dot\gamma^\mu(t)$$

Since $\gamma\colon [\sigma,\tau] \to M$, we get equation (4.65) from here, for $t = \sigma$, with $\Gamma^a_{b\mu}(x) = -\frac{\partial A^a_b(x)}{\partial x^\mu}$ and $G^a_\mu(x) = \frac{\partial A^a(x)}{\partial x^\mu}$ for $x \in \pi(U)$. □

Proposition 4.8. *There is a bijective mapping α between the sets of affine connections and of pairs of a linear connection and a linear mapping $G\colon \mathcal{X}(M) \to \mathrm{End}(\mathrm{Sec}((E,\pi,M)^*))$.*

Proof. If ${}^A\Delta^h$ is an affine connection with 2-index coefficients give by (4.65) (see theorem 4.3), then (see the discussion after equation (4.65)) to it corresponds the pair $\alpha({}^A\Delta^h) := ({}^L\Delta^h, G)$ of a linear connection, with 3-index coefficients $\Gamma^a_{b\mu}$ and linear mapping $G\colon \mathcal{X}(M) \to \mathrm{End}(\mathrm{Sec}((E,\pi,M)^*))$, with components G^a_μ. Conversely, to a pair $({}^L\Delta^h, G)$, locally described via the 3-index coefficients $\Gamma^a_{b\mu}$ of ${}^L\Delta^h$ and components G^a_μ of G, there corresponds an affine connection ${}^A\Delta^h = \alpha^{-1}({}^L\Delta^h, G)$ with 2-index coefficients given by (4.65). □

In Subsect. 4.4, it was demonstrated that covariant derivatives can be introduced for affine connections, not only for linear ones.

Proposition 4.9. *The covariant derivative for an affine connection ${}^A\Delta^h$ coincides with the one for the linear connection ${}^L\Delta^h$ given via $\alpha({}^A\Delta^h) = ({}^L\Delta^h, G)$ with α defined in the proof of proposition 4.8.*

Proof. Apply (4.38)–(4.45). □

If a linear connection ${}^{L}\Delta^h$ and an affine one ${}^{A}\Delta^h$ are connected by $\alpha({}^{A}\Delta^h) = ({}^{L}\Delta^h, G)$ for some G, then some of their characteristics coincide; e.g. such are their fibre coefficients (see (3.37b), (4.65) and (4.22)) and all quantities expressed via the corresponding to them (identical) covariant derivatives. However, quantities, containing (depending on) partial derivatives relative to the basic coordinates $\{u^\mu\}$, are generally different for those connections. For instance, if ${}^{A}R^a_{\mu\nu}$ and ${}^{L}R^a_{\mu\nu}$ are the fibre components of the curvatures of ${}^{A}\Delta^h$ and ${}^{L}\Delta^h$, respectively, then, by (3.37a) and (4.65), we have

$$ {}^{A}R^a_{\mu\nu} = -({}^{L}R^a_{b\mu\nu} \circ \pi) \cdot u^b - T^a_{\mu\nu} \circ \pi \tag{4.72} $$

$$ {}^{L}R^a_{\mu\nu} = -({}^{L}R^a_{b\mu\nu} \circ \pi) \cdot u^b \tag{4.73} $$

where (see (4.34))

$$ {}^{L}R^a_{b\mu\nu} := \frac{\partial}{\partial x^\mu}(\Gamma^a_{b\nu}) - \frac{\partial}{\partial x^\nu}(\Gamma^a_{b\mu}) - \Gamma^c_{b\mu}\Gamma^a_{c\nu} + \Gamma^c_{b\nu}\Gamma^a_{c\mu}, \tag{4.74} $$

$$ T^a_{\mu\nu} := -\frac{\partial}{\partial x^\mu}(G^a_\nu) + \frac{\partial}{\partial x^\nu}(G^a_\mu) + \Gamma^a_{c\nu}G^c_\mu - \Gamma^a_{c\mu}G^c_\nu \tag{4.75} $$

and the functions $T^a_{\mu\nu}$ have a sense of components of the torsion of ${}^{L}\Delta^h$ relative to G [21, pp. 42, 46].

Thus, in general, the affine connections and linear connections are essentially different. However, they imply identical theories of covariant derivatives.

If, for some reason, the linear mapping G is fixed, then the set of linear connections $\{{}^{L}\Delta^h\}$ can be identified with the subset $\{\alpha^{-1}({}^{L}\Delta^h, G)\}$ of the set of affine connections $\{{}^{A}\Delta^h\}$. We shall exemplify this situation on the tangent bundle $(T(M), \pi_T, M)$ over a manifold M. Using the base indices $\mu, \nu, \dots$ for the fibre ones $a, b, \dots$ according to the rule $a \mapsto \mu = a - \dim M$ (see Subsect. 4.2), we rewrite (4.65) as

$$ \Gamma^\mu_\nu = -(\Gamma^\mu_{\lambda\nu} \circ \pi_T) \cdot u_1^\lambda + G^\mu_\nu \circ \pi_T. \tag{4.76} $$

Now the affine connections on $(T(M), \pi_T, M)$ are the *generalized affine connections on* M [2, ch. III, § 3]. The choice of G via

$$ G^\mu_\nu \colon M \to \delta^\mu_\nu, \tag{4.77} $$

which corresponds to the identical transformation of the spaces tangent to M, singles out the set of *affine connections on* M – see [2, ch. III, § 3] or [23, pp. 103–105] – (known also as *Cartan connections on* M [21, p. 46]) whose 2-index coefficients have the form (see (4.76), (4.19a) and (4.77))

$$ \Gamma^\mu_\nu = -(\Gamma^\mu_{\lambda\nu} \circ \pi_T) \cdot \mathrm{d}x^\lambda + \delta^\mu_\nu. \tag{4.78} $$

Combining this with proposition 4.8, we derive

Proposition 4.10 (cf. [2, ch. III, § 3, theorem 3.3])**.** *There is a bijective correspondence between the sets of linear connections and of affine ones on a manifold.*

Often the terms “linear connection” and “affine connection” on a manifold are used as synonyms, due to the last result.

5. Morphisms of bundles with connection [18]

A *morphism* between two bundles (E, π, M) and (E', π', M') is a pair of mappings (F, f) such that $F \colon E \to E'$, $f \colon M \to M'$, and $\pi' \circ F = f \circ \pi$. If (U, u) and (U', u') are charts in E

[18] Some ideas in this section are borrowed from [20, ch. I, § 6]

and E', respectively, and $F(U) \subseteq U'$, we have the following local representation of (F, f)

$$\bar{F} = u' \circ F \circ u^{-1} \colon u(U) \to u'(U') \tag{5.1a}$$
$$\bar{f} = x' \circ f \circ x^{-1} \colon x(V) \to x'(V'), \tag{5.1b}$$

where (V, x) and (V', x') are local charts respectively on M and M'. Further, we assume that $U' = F(U)$ and that the charts in the base and bundle spaces respect the fibre structure, $V = \pi(U)$ and $V' = \pi'(U')$ so that $V' = f(V)$, and that the basic coordinates are $u^\mu = x^\mu \circ \pi$ and $u'^{\mu'} = x'^{\mu'} \circ \pi'$. Here and henceforth the quantities referring to (E', π', M') will inherit the same notation as the similar ones with respect to (E, π, M) with exception of the prime symbol added to the latter ones; in particular, the primed indices $\lambda', \mu', \nu', \dots$ and $a', b', c', \dots$ run respectively over the ranges $1, \dots, n' = \dim M'$ and $n'+1, \dots, n'+r' = \dim E'$ with r' being the fibre dimension of (E', π', M'), i.e. $r' = \dim((\pi')^{-1}(p')$ for $p' \in M'$.

Using the local coordinates $\{x^\mu\}$ on M and $\{u^\mu = x^\mu \circ \pi, u^a\}$ on E, we rewrite (5.1) as (cf. (3.1))

$$\bar{F}^{I'} = u'^{I'} \circ F \circ u^{-1} \colon u(U) \to \mathbb{K} \tag{5.1$'$a}$$
$$\bar{f}^{\mu'} = x'^{\mu'} \circ f \circ x^{-1} \colon x(\pi(U)) \to \mathbb{K}, \tag{5.1$'$b}$$

i.e. one can simply write $u'^{I'} = \bar{F}^{I'}(u^1, \dots, u^{n+r})$ and $x'^{\mu'} = \bar{f}^{\mu'}(x^1, \dots, x^n)$. However, in what follows, the mappings

$$F^{\mu'} := u'^{\mu'} \circ F = x'^{\mu'} \circ \pi \circ F = x'^{\mu'} \circ f \circ \pi \colon U \to \mathbb{K} \quad F^{a'} := u'^{a'} \circ F \colon U \to \mathbb{K} \tag{5.3a}$$
$$f^{\mu'} := x'^{\mu'} \circ f \colon \pi(U) \to \mathbb{K} \tag{5.3b}$$

will be employed. The reason is that the derivatives

$$F^{I'}_{,J} := \frac{\partial}{\partial u^J} F^{I'} \colon p \mapsto \frac{\partial}{\partial u^J}\Big|_p F^{I'} = \frac{\partial (F^{I'} \circ u^{-1})}{\partial (u^J \circ u^{-1})}\Big|_{u(p)} = \frac{\partial \bar{F}^{I'}}{\partial (u^J \circ u^{-1})}\Big|_{u(p)} \qquad p \in U$$

(note, $\{u^J \circ u^{-1}\}$ are Cartesian coordinates on $u(U) \subseteq \mathbb{K}^{n+r}$) are the elements of the matrix of the tangent mapping $F_* \colon T(E) \to T(E')$ in the charts (U, u) and (U', u'). Indeed, since this matrix, known as the Jacobi matrix of F, is defined by [5, sec. 1.23(a)]

$$F_*(\partial_J|_p) = F^{I'}_J|_p (\partial'_{I'}|_{F(p)}) \qquad p \in U, \tag{5.4}$$

we have

$$[F^{I'}_J]_{\text{in } (\{\partial_K\}, \{\partial_{K'}\})} = \left[\frac{\partial F^{I'}}{\partial u^J}\right]_{\substack{I'=1,\dots,n'+r' \\ J=1,\dots,n+r}} = \begin{pmatrix} [F^{\nu'}_{,\mu}] & 0_{n' \times r} \\ [F^{a'}_{,\mu}] & [F^{a'}_{,b}] \end{pmatrix} = \begin{bmatrix} F^{\nu'}_{,\mu} & 0_{n' \times r} \\ F^{a'}_{,\mu} & F^{a'}_{,b} \end{bmatrix} . \tag{5.5}$$

Let connections Δ^h and Δ'^h on (E, π, M) and $(E,' \pi', M')$, respectively, be given. To the local coordinates $\{u^I\}$ and $\{u'^I\}$ correspond the adapted frames (see (3.27)–(3.30))

$$\begin{gathered} (X_\mu, X_a) = (\partial_\nu, \partial_b) \cdot \begin{bmatrix} \delta^\nu_\mu & 0 \\ +\Gamma^b_{\ \mu} & \delta^b_a \end{bmatrix} = (\partial_\mu + \Gamma^b_{\ \mu} \partial_b, \partial_a) \\ (X'_{\mu'}, X'_{a'}) = (\partial'_{\nu'}, \partial'_{b'}) \cdot \begin{bmatrix} \delta^{\nu'}_{\mu'} & 0 \\ +\Gamma^{b'}_{\ \mu'} & \delta^{b'}_{a'} \end{bmatrix} = (\partial'_{\mu'} + \Gamma'^{b'}_{\ \mu'} \partial_{b'}, \partial_{a'}), \end{gathered} \tag{5.6}$$

where $\partial_I := \frac{\partial}{\partial u^I}$, and adapted coframes

$$\begin{pmatrix} \omega^\mu \\ \omega^a \end{pmatrix} = \begin{bmatrix} \delta^\mu_\nu & 0 \\ -\Gamma^a_{\ \nu} & \delta^a_b \end{bmatrix} \cdot \begin{pmatrix} \mathrm{d}u^\nu \\ \mathrm{d}u^b \end{pmatrix} = \dots \qquad \begin{pmatrix} \omega'^{\mu'} \\ \omega'^{a'} \end{pmatrix} = \begin{bmatrix} \delta^{\mu'}_{\nu'} & 0 \\ -\Gamma'^{a'}_{\ \nu'} & \delta^{a'}_{b'} \end{bmatrix} \cdot \begin{pmatrix} \mathrm{d}u'^{\nu'} \\ \mathrm{d}u'^{b'} \end{pmatrix} = \dots . \tag{5.7}$$

The symbols Γ^a_μ and $\Gamma'^{a'}_{\mu'}$ in (5.6) and (5.7) denote the 2-index coefficients of respectively Δ^h and Δ'^h in the respective adapted frames.

If $\{e_I\}$ and $\{e'_{I'}\}$ are arbitrary frames over U in $T(E)$ and over $U' = F(U)$ in $T(E')$, respectively, the (Jacobi) matrix of F_* in them is defined via (cf. (5.4))

$$F_*(e_I|_p) = (F^{I'}_I|_p)(e'_{I'}|_{F(p)}). \tag{5.8}$$

In particular, in the adapted frames (5.6), we have $F_*(X_I|_p) = (F^{I'}_I|_p)(X'_{I'}|_{F(p)})$ and therefore the Jacobi matrix of F_* in the adapted frames (5.6) is [19]

$$\begin{aligned}[][F^{I'}_J] = [F^{I'}_J]_{\text{in }(\{X_K\},\{X'_{K'}\})} = \begin{bmatrix} F^{\mu'}_\nu & F^{\mu'}_a \\ F^{b'}_\nu & F^{b'}_a \end{bmatrix} &= \begin{bmatrix} \delta^{\mu'}_{\lambda'} & 0 \\ -\Gamma'^{b'}_{\lambda'} \circ F & \delta^{b'}_{c'} \end{bmatrix} \cdot \begin{bmatrix} F^{\lambda'}_{,\varrho} & 0 \\ F^{c'}_{,\varrho} & F^{c'}_{,d} \end{bmatrix} \cdot \begin{bmatrix} \delta^\varrho_\nu & 0 \\ +\Gamma^d_\nu & \delta^d_a \end{bmatrix} \\ &= \begin{bmatrix} F^{\mu'}_{,\nu} & 0 \\ X_\nu(F^{b'}) - (\Gamma'^{b'}_{\lambda'} \circ F)F^{\lambda'}_{,\nu} & F^{b'}_{,a} \end{bmatrix}\end{aligned} \tag{5.9}$$

with $F^{I'}_{,J} := \frac{\partial F^{I'}}{\partial u^J}$ defining the matrix of F_* in $(\{\partial_K\},\{\partial'_{K'}\})$ via (5.5). Thus, the general formula (5.8) now reads

$$F_*(X_\mu, X_a) = (X'_{\nu'}, X'_{b'}) \cdot \begin{bmatrix} F^{\nu'}_\mu & 0 \\ F^{b'}_\mu & F^{b'}_a \end{bmatrix} = (F^{\nu'}_{,\mu} X'_{\nu'} + F^{b'}_\mu X'_{b'}, F^{b'}_{,a} X'_{b'}) \tag{5.10}$$

with

$$F^{b'}_\mu = X_\mu(F^{b'}) - (\Gamma'^{b'}_{\lambda'} \circ F) \cdot F^{\lambda'}_{,\mu}. \tag{5.11}$$

From (5.8), it is clear that the elements $F^{I'}_J|_p$ of the Jacobi matrix of F_* at $p \in U$ are elements of a $(1,1)$ (mixed) tensor from $T^*_p(E) \otimes T_{F(p)}(E')$; in particular, if the adapted frames are changed (see (3.31)), the block structure of (5.9) is preserved and the elements of its blocks are transformed as elements of the corresponding to them tensors [20]. An important corollary from (5.10) is

$$F_*(\Delta^h) \subseteq \Delta'^h \iff F^{b'}_\mu = 0 \tag{5.12}$$

in any pair $(\{X_I\},\{X_{I'}\})$ of adapted frames. If it happens that $F_*(\Delta^h) = \Delta'^h$, we say that *$F$ preserves the connection Δ^h*, i.e. F is a *connection preserving mapping*; in particular, if $(E',\pi',M') = (E,\pi,M)$ and $F_*(\Delta^h) = \Delta'^h$, the mapping F is called a *symmetry* of Δ^h.

If the bundles considered are vectorial ones, the fibre coordinates, morphisms, and connections which are compatible with the vector structure must be linear on the fibres, viz.

$$F^{a'} = (\mathcal{F}^{a'}_b \circ \pi)u^b \quad \Gamma^a_\mu = -(\Gamma^a_{b\mu} \circ \pi)u^b \quad \Gamma^{a'}_{\mu'} = -(\Gamma^{a'}_{b'\mu'} \circ \pi')u^{b'}, \tag{5.13}$$

where the functions $\mathcal{F}^{a'}_b \colon \pi(U) \to \mathbb{K}$ are of class C^1 and $\Gamma^a_{b\mu}$ (resp. $\Gamma^{a'}_{b'\mu'}$) are the 3-index coefficients of the linear connection Δ^h (resp. Δ'^h). Consequently, in a case of vector bundles, the Jacobi matrix (5.9) takes the form

$$\begin{bmatrix} F^{\mu'}_\nu & F^{\mu'}_a \\ F^{b'}_\nu & F^{b'}_a \end{bmatrix} = \begin{bmatrix} F^{\mu'}_{,\nu} & 0 \\ (F^{b'}_{c\nu} \circ \pi)u^c & \mathcal{F}^{b'}_a \circ \pi \end{bmatrix} \tag{5.14}$$

with

$$F^{b'}_{a\mu} := \partial_\mu(\mathcal{F}^{b'}_a) - \Gamma^c_{a\mu}\mathcal{F}^{b'}_c + (\Gamma'^{b'}_{c'\lambda'} \circ f) \cdot \mathcal{F}^{c'}_a \cdot f^{\lambda'}_\mu, \tag{5.15}$$

[19] The changes $e := \{e_I\} \mapsto \{B^J_I e_J\}$ and $e' := \{e'_{I'}\} \mapsto \{B'^{J'}_{I'} e'_{J'}\}$, with non-degenerate matrix-valued functions $B := [B^J_I]$ and $B' := [B'^{J'}_{I'}]$, imply the transformation $F_{(e,e')} := [F^{I'}_J] \mapsto (B')^{-1} \cdot F_{(e,e')} \cdot B$ of the Jacobi matrix of F_*. From here, (5.9) follows immediately.

[20] E.g. $F^{b'}_\nu(p)$ are elements of a tensor from the tensor space spanned by $\{\omega^\nu|_p \otimes X'_{b'}|_{F(p)}\}$.

where we have used that $\pi' \circ F = f \circ \pi$ and $u'^{c'} \circ F = F^{c'}$ and we have set $f^{\lambda'}_{\mu} := \frac{\partial(x'^{\lambda'} \circ f)}{\partial x^{\mu}}$, so that $F^{\lambda'}_{,\mu} = f^{\lambda'}_{\mu} \circ \pi$. Therefore (5.11) now reads

$$F^{b'}_{\mu} = (F^{b'}_{a\mu} \circ \pi) \cdot u^{a}. \tag{5.16}$$

If M and M' are manifolds and $f\colon M \to M'$ is of class C^1, the above general considerations are valid for the morphism (f_*, f) of the tangent bundles $(T(M), \pi_T, M)$ and $(T(M'), \pi'_T, M')$. A peculiarity of a tangent bundle is that the fibre dimension of the bundle equals to the dimension of its base. Due to that fact, the base indices $\lambda, \mu, \nu, \dots = 1, \dots, n$ is convenient to be used for the fibre ones $a, b, c, \dots = n+1, \dots, n+r$ according to the rule

$$a \mapsto \mu = a - \dim M, \tag{5.17a}$$

which must be combined with a change of the notation for the fibre coordinates, like

$$u^{a} \mapsto u^{\mu}_{1}, \tag{5.17b}$$

as otherwise the change (5.17a) will entail $u^a \mapsto u^\mu$, the result of which coincides with the notation for the basic coordinates. [21] Since $f_*\bigl(\frac{\partial}{\partial x^{\mu}}\big|_{z}\bigr) = \frac{\partial(x'^{\mu'} \circ f)}{\partial x^{\mu}}\big|_{z} \frac{\partial}{\partial x'^{\mu'}}\big|_{f(z)}$ for $z \in \pi(U)$ [5, sec. 1.23(a)], the Jacobi matrix of f relative to the charts $(\pi(U), x)$ and $(\pi'(U'), x') = (f(\pi(U)), x')$ has the elements

$$f^{\mu'}_{\nu} := \frac{\partial(x'^{\mu'} \circ f)}{\partial x^{\nu}} \colon \pi(U) \to \mathbb{K}. \tag{5.18}$$

Combining this with the definition of the vector fibre coordinates u^{μ}_{1}, $u^{\mu}_{1}\bigl(p^{\nu}\frac{\partial}{\partial x^{\nu}}\big|_{\pi(p)}\bigr) = p^{\mu}$, we see that (5.3), with f_* for F, reads

$$u'^{\mu'} = f^{\mu'}_{*}(u^{1}, \dots, u^{n}) = x'^{\mu'} \circ f \circ \pi \quad u'^{\mu'}_{1} = f'^{\mu'}_{*}(u^{1}, \dots, u^{n}, u^{1}_{1}, \dots, u^{n}_{1}) = (f^{\mu'}_{\nu} \circ \pi) \cdot u^{\nu}_{1} \tag{5.19a}$$

$$x'^{\mu'} = f^{\mu'}(x^{1}, \dots, x^{n}) = x^{\mu'} \circ f. \tag{5.19b}$$

Therefore the derivatives in (5.5) and (5.9)–(5.11) should be replace according to $(u^{\mu} = x^{\mu} \circ \pi)$

$$F^{\nu'}_{,\mu} \mapsto \frac{\partial f^{\mu'}_{*}}{\partial x^{\mu}} = f^{\nu'}_{\mu} \circ \pi \quad F^{a'}_{,\nu} \mapsto \frac{\partial f^{\mu'}_{*}}{\partial x^{\nu}} = \Bigl(\frac{\partial f^{\mu'}_{\lambda}}{\partial x^{\nu}} \circ \pi\Bigr) u^{\lambda}_{1} \quad F^{a'}_{,b} \mapsto \frac{\partial f^{\mu'}_{*}}{\partial u^{\nu}_{1}} = f^{\mu'}_{\nu} \circ \pi. \tag{5.20}$$

If Δ^h and Δ'^h are linear connections on M and M', respectively, their 2- and 3-index coefficients are connected through (cf. (5.13))

$$\Gamma^{\lambda}_{\nu} = -(\Gamma^{\lambda}_{\mu\nu} \circ \pi) \cdot u^{\lambda}_{1} \quad \Gamma'^{\lambda'}_{\nu'} = -(\Gamma'^{\lambda'}_{\mu'\nu'} \circ \pi') \cdot u^{\lambda'}_{1}. \tag{5.21}$$

Thus the Jacobi matrix of $(f_*)_* =: f_{**}$ in the pair of frames $\bigl(\{X_{\mu} = \frac{\partial}{\partial x^{\mu}} + \Gamma^{\lambda}_{\mu}\frac{\partial}{\partial u^{\lambda}_{1}}, X^{1}_{\nu} = \frac{\partial}{\partial u^{\nu}_{1}}\}, \{X'_{\mu'}, X'^{1}_{\nu'}\}\bigr)$ is (cf. (5.14) and (5.15))

$$\begin{bmatrix} f^{\mu'}_{\nu} \circ \pi & 0 \\ -f^{\varrho'}_{\lambda\nu} \circ \pi) \cdot u^{\lambda}_{1} & f^{\varrho'}_{\tau} \circ \pi \end{bmatrix} \tag{5.22}$$

[21] The subscript 1 in (5.17b) indicates that u^{μ}_{1} are fibre coordinates in the *first* order tangent bundle $(T(M), \pi_T, M)$ over M.

where

$$f^{\lambda'}_{\mu\nu} := f^{\lambda'}_{*\mu\nu} := \partial_\nu(f^{\lambda'}_{\mu}) - f^{\lambda'}_{\sigma}\Gamma^{\sigma}_{\mu\nu} + (\Gamma'^{\lambda'}_{\sigma'\tau'} \circ f) f^{\sigma'}_{\mu} f^{\tau'}_{\nu}. \tag{5.23}$$

The quantities (5.23) are components of a $T(M')$-valued 2-form on M, i.e. of an element in $T(M') \otimes \Lambda^2(M)$. [22]

6. General (co)frames

Until now two special kinds of local (co)frames in the (co)tangent bundle to the bundle space of a bundle were employed, viz. the natural holonomic ones, induced by some local coordinates, and the adapted (co)frames determined by local coordinates and a connection on the bundle. The present section is devoted to (re)formulation of some important results and formulae in arbitrary (co)frames, which in particular can be natural or adapted (if a connection is presented) ones.

Let (E, π, M) be a C^2 bundle and $\{e_I\}$ a (local) frame in $T(E)$. The components C^K_{IJ} of the anholonomy object of $\{e_I\}$ are defined by (3.19) and a change

$$\{e_I\} \mapsto \{\bar{e}_I = B^J_I e_J\} \tag{6.1}$$

with a non-degenerate matrix-valued function $B = [B^J_I]^{n+r}_{I,J=1}$ entails (see (2.9))

$$C^K_{IJ} \mapsto \bar{C}^K_{IJ} = (B^{-1})^K_L \big(B^M_I e_M(B^L_J) - B^M_J e_M(B^L_I) + B^M_I B^N_J C^L_{MN} \big). \tag{6.2}$$

Let a connection Δ^h on (E, π, M) be given. If $\{e_I\}$ is a specialized frame for Δ^h (see Sebsect. 3.2), then the set $\{C^K_{IJ}\}$ is naturally divided into the six groups (3.20). The value of that division is in its invariance with respect to the class of specialized frames, which means that, if $\{\bar{e}_I\}$ is also a specialized frame, then the transformed components of the elements of each group are functions only in the elements of the non-transformed components of the same group — see (3.24), (3.21), and (2.9). By means of (6.1), one can prove that, if such a division holds in a frame $\{e_I\}$, then it holds in $\{\bar{e}_I\}$ if and only if the matrix-valued function B is of the form (3.16). In particular, we cannot talk about fibre coefficients of Δ^h and of fibre components of the curvature of Δ^h in frames more general than the specialized ones as in that case the transformation (6.1), with $\{e_I\}$ (resp. $\{\bar{e}_I\}$) being a specialized (resp. non-specialized) frame, will mix, for instance, the fibre coefficients and the curvature's fibre components of Δ^h in $\{\bar{e}_I\}$ — see (6.2).

It is a simple, but important, fact that the specialized frames are (up to renumbering) the most general ones which respect the splitting of $T(E)$ into vertical and horizontal components. Suppose $\{e_I\}$ is a specialized frame. Then the general element of the set of all specialized frames is (see (3.4a) and (3.16))

$$(\bar{e}_\mu, \bar{e}_a) = (e_\nu, e_b) \cdot \begin{bmatrix} A^\nu_\mu & 0 \\ 0 & A^b_a \end{bmatrix} = (A^\nu_\mu e_\nu, A^b_a e_b), \tag{6.3a}$$

where $[A^\nu_\mu]^n_{\mu,\nu=1}$ and $[A^b_a]^{n+r}_{a,b=n+1}$ are non-degenerate matrix-valued functions on E, which are constant on the fibres of (E, π, M), i.e. we can set $A^\nu_\mu = B^\nu_\mu \circ \pi$ and $A^b_a = B^b_a \circ \pi$ for some non-degenerate matrix-valued functions $[B^\nu_\mu]$ and $[B^b_a]$ on M. Respectively, the general specialized coframe dual to $\{\bar{e}_I\}$ is (see (3.4b) and (3.16))

$$\begin{pmatrix} \bar{e}^\mu \\ \bar{e}^a \end{pmatrix} = \begin{bmatrix} [A^\lambda_\rho]^{-1} & 0 \\ 0 & [A^c_d]^{-1} \end{bmatrix} \cdot \begin{pmatrix} e^\nu \\ e^b \end{pmatrix} = \begin{bmatrix} \big([A^\lambda_\rho]^{-1}\big)^\mu_\nu e^\nu \\ \big([A^c_d]^{-1}\big)^a_b e^b \end{bmatrix}, \tag{6.3b}$$

[22] Moreover, if we consider $f^{\mu'}_\nu$, defined via (5.18), as components of an element in $T_{f(p)}(M') \otimes \Lambda^1_p(M)$, then (5.23) are the components of the *mixed* covariant derivative (along $\frac{\partial}{\partial x^\nu}$) of $f^{\mu'}_\mu (\partial'_{\mu'}|_{f(\cdot)}) \otimes \mathrm{d}u^\mu$ relative to the connection $\Delta^h \times \Delta'^h$ on $M \times M$.

where $\{e^I\}$ is the specialized coframe dual to $\{e_I\}$, $e^I(e_J) = \delta^I_J$.

Since $\pi_*|_{\Delta^h}: \{X \in \Delta^h\} \to \mathcal{X}(M)$ is an isomorphism, any basis $\{\varepsilon_\mu\}$ *for* Δ^h defines a basis $\{E_\mu\}$ of $\mathcal{X}(M)$ such that

$$E_\mu = \pi_*|_{\Delta^h}(\varepsilon_\mu) \tag{6.4}$$

and v.v., a basis $\{E_\mu\}$ for $\mathcal{X}(M)$ induces a basis $\{\varepsilon_\mu\}$ for Δ^h via

$$\varepsilon_\mu = (\pi_*|_{\Delta^h})^{-1}(E_\mu). \tag{6.5}$$

Similarly, there is a bijection $\{\varepsilon^\mu\} \mapsto \{E^\mu\}$ between the 'horizontal' coframes $\{\varepsilon^\mu\}$ and the coframes $\{E^\mu\}$ dual to the frames in $T(M)$ ($E^\mu \in \Lambda^1(M)$, $E^\mu(E_\nu) = \delta^\mu_\nu$). Thus a 'horizontal' change

$$\varepsilon_\mu \mapsto \bar{\varepsilon}_\mu = (B^\nu_\mu \circ \pi)\varepsilon_\nu, \tag{6.6}$$

which is independent of a 'vertical' one given by

$$\varepsilon_a \mapsto \bar{\varepsilon}_a = (B^b_a \circ \pi)\varepsilon_b \tag{6.7}$$

with $\{\varepsilon_a\}$ being a basis for Δ^v, is equivalent to the transformation

$$E_\mu \mapsto \bar{E}_\mu = B^\nu_\mu E_\nu \tag{6.8}$$

of the basis $\{E_\mu\}$ for $\mathcal{X}(M)$, related via (6.4) to the basis $\{\varepsilon_\mu\}$ for Δ^h. Here $[B^\nu_\mu]$ and $[B^b_a]$ are non-degenerate matrix-valued functions on M.

As $\pi_*(\varepsilon_a) = 0 \in \mathcal{X}(M)$, the 'vertical' transformations (6.7) do not admit interpretation analogous to the 'horizontal' ones (6.6). However, in a case of a *vector* bundle (E, π, M), they are tantamount to changes of frames in the bundle space E, i.e. of the bases for $\mathrm{Sec}(E, \pi, M)$. Indeed, if v is the mapping defined by (4.3), the sections

$$E_a = v^{-1}(\varepsilon_a) \tag{6.9}$$

form a basis for $\mathrm{Sec}(E, \pi, M)$ as the vertical vector fields ε_a form a basis for Δ^v. Conversely, any basis $\{E_a\}$ for the sections of (E, π, M) induces a basis $\{\varepsilon_a\}$ for Δ^v such that

$$\varepsilon_a = v(E_a). \tag{6.10}$$

As v and v^{-1} are linear, the change (6.7) is equivalent to the transformation

$$E_a \mapsto \bar{E}_a = B^b_a E_b \tag{6.11}$$

of the frame $\{E_a\}$ in E related to $\{\varepsilon_a\}$ via (6.9). In this way, we see that *any specialized frame* $\{\varepsilon_I\} = \{\varepsilon_\mu, \varepsilon_a\}$ *for a connection on a vector bundle* (E, π, M) *is equivalent to a pair of frames* $(\{E_\mu\}, \{E_a\})$ such that $\{E_\mu\}$ is a basis for the set $\mathcal{X}(M)$ of vector fields on the base M, i.e. for the sections of the tangent bundle $(T(M), \pi_T, M)$ (and hence is a frame in $T(M)$ over M), and $\{E_a\}$ is a basis for the set $\mathrm{Sec}(E, \pi, M)$ of sections of the initial bundle (and hence is a frame in E over M). Since conceptually the frames in $T(M)$ and E are easier to be understood and in some cases have a direct physical interpretation, one often works with the pair of frames $(\{E_\mu = \pi_*|_{\Delta^h}(\varepsilon_\mu)\}, \{E_a = v^{-1}(\varepsilon_a)\})$ instead with a specialized frame $\{\varepsilon_I\} = \{\varepsilon_\mu, \varepsilon_a\}$; for instance $\{E_\mu\}$ and $\{E_a\}$ can be completely arbitrary frames in $T(M)$ and E, respectively, while the specialized frames represent only a particular class of frames in $T(E)$.

One can *mutatis mutandis* localize the above considerations when M is replaced with an open subset U_M in M and E is replaced with $U = \pi^{-1}(U_M)$. Such a localization is important

when the bases/frames considered are connected with some local coordinates or when they should be smooth.[23]

Let us turn now our attention to frames adapted to local coordinates $\{u^I\}$ on an open set $U \subseteq E$ for a given connection Δ^h on a general C^1 bundle (E, π, M) (see (3.27)–(3.30)). Since in their definition the local coordinates $\{u^I\}$ enter only via the vector fields $\partial_I := \frac{\partial}{\partial u^I} \in \mathcal{X}(E)$, we can generalize this definition by replacing $\{\partial_I\}$ with an arbitrary frame $\{e_I\}$ defined in $T(E)$ over an open set $U \subseteq E$ and such that $\{e_a|_p\}$ is a *basis for the space* $T_p(\pi^{-1}(\pi(p)))$ *tangent to the fibre through* $p \in U$. So, using $\{e_I\}$ for $\{\partial_I\}$, we have

$$(e^U_\mu, e^U_a) = (D^\nu_\mu e_\mu + D^a_\mu e_a, D^b_a e_b) = (e_\nu, e_b) \cdot \begin{pmatrix} [D^\nu_\mu] & 0 \\ [D^b_\mu] & [D^b_a] \end{pmatrix}, \tag{6.12}$$

where $\{e^U_I\}$ is a *specialized* frame in $T(U)$, $[D^\nu_\mu]$ and $[D^b_a]$ are non-degenerate matrix-valued functions on U, and $D^a_\mu \colon U \to \mathbb{K}$.

Definition 6.1. The specialized frame $\{X_I\}$ over U in $T(U)$, obtained from (6.12) via an admissible transformation (3.4a) with matrix $A = \begin{pmatrix} [D^\mu_\nu]^{-1} & 0 \\ 0 & [D^a_b]^{-1} \end{pmatrix}$, is called *adapted to the frame* $\{e_I\}$ *for* Δ^h. [24]

Exercise 6.1. Using (3.4) and (3.16), verify that the adapted frame $\{X_I\}$ and coframe $\{\omega^I\}$ dual to it are independent of the particular specialized frame $\{e^U_I\}$ entering into their definitions via (6.12). The equalities (6.13a) and (6.21) derived below are indirect proof of that fact too.

According to (3.4), the adapted frame $\{X_I\} = \{X_\mu, X_a\}$ and the dual to it coframe $\{\omega^I\} = \{\omega^\mu, \omega^a\}$ are given by the equations

$$(X_\mu, X_a) = (e_\nu, e_b) \cdot \begin{bmatrix} \delta^\nu_\mu & 0 \\ +\Gamma^b_\mu & \delta^b_a \end{bmatrix} = (e_\mu + \Gamma^b_\mu e_b, e_a) \tag{6.13a}$$

$$\begin{pmatrix} \omega^\mu \\ \omega^a \end{pmatrix} = \begin{bmatrix} \delta^\mu_\nu & 0 \\ -\Gamma^a_\nu & \delta^a_b \end{bmatrix} \cdot \begin{pmatrix} e^\nu \\ e^b \end{pmatrix} = \begin{pmatrix} e^\mu \\ e^a - \Gamma^a_\nu e^\nu \end{pmatrix}, \tag{6.13b}$$

where $\{e^I\}$ is the coframe dual to $\{e_I\}$, $e^I(e_J) = \delta^I_J$, and the functions $\Gamma^a_\mu \colon U \to \mathbb{K}$, called *(2-index) coefficients of* Δ^h in $\{X_I\}$, are defined by

$$[\Gamma^a_\mu] := +[D^a_\nu] \cdot [D^\nu_\mu]^{-1}. \tag{6.14}$$

Proposition 6.1. *A change* $\{e_I\} \mapsto \{\tilde{e}_I\}$ *with*

$$(\tilde{e}_\mu, \tilde{e}_a) = (e_\nu, e_b) \cdot \begin{pmatrix} [A^\nu_\mu] & 0 \\ [A^b_\mu] & [A^b_a] \end{pmatrix} = (A^\nu_\mu e_\nu + A^b_\mu e_b, A^b_a e_b), \tag{6.15}$$

where $[A^\nu_\mu]$ *and* $[A^b_a]$ *are non-degenerate matrix-valued functions on* U*, which are constant on the fibres of* (E, π, M)*, and* $A^b_\mu \colon U \to \mathbb{K}$*, entails the transformations*

$$(X_\mu, X_a) \mapsto (\tilde{X}_\mu, \tilde{X}_a) = (\tilde{e}_\mu + \tilde{\Gamma}^b_\mu \tilde{e}_b, \tilde{e}_a) = (A^\nu_\mu X_\nu, A^b_a X_b) = (X_\nu, X_b) \cdot \begin{bmatrix} A^\nu_\mu & 0 \\ 0 & A^b_a \end{bmatrix} \tag{6.16}$$

$$\Gamma^a_\mu \mapsto \tilde{\Gamma}^a_\mu = \left([A^c_d]^{-1}\right)^a_b (\Gamma^b_\nu A^\nu_\mu - A^b_\mu) \tag{6.17}$$

of the frame $\{X_I\}$ *adapted to* $\{e_I\}$ *and of the coefficients* Γ^a_μ *of* Δ^h *in* $\{X_I\}$*, i.e.* $\{\tilde{X}_I\}$ *is the frame adapted to* $\{\tilde{e}_I\}$ *and* $\tilde{\Gamma}^a_\mu$ *are the coefficients of* Δ^h *in* $\{\tilde{X}_I\}$*.*

[23] Recall, not every manifold admits a *global* nowhere vanishing C^m, $m \geq 0$, vector field (see [26] or [31, sec. 4.24]); e.g. such are the even-dimensional spheres $\mathbb{S}^{2k}$, $k \in \mathbb{N}$, in Euclidean space.

[24] Recall, here and below the adapted frames are defined only with respect to frames $\{e_I\} = \{e_\mu, e_a\}$ such that $\{e_a\}$ is a basis for the vertical distribution Δ^v over U, i.e. $\{e_a|_p\}$ is a basis for Δ^v_p for all $p \in U$. Since Δ^v is integrable, the relation $e_a \in \Delta^v$ for all $a = n+1, \dots, n+r$ implies $[e_a, e_b]_- \in \Delta^v$ for all $a, b = n+1, \dots, n+r$.

Proof. Apply (6.12)–(6.14). □

Note 6.1. If $\{e_I\}$ and $\{\tilde{e}_I\}$ are adapted, then $A^b_\mu = 0$. If $\{Y_I\}$ is a specialized frame, it is adapted to any frame $\{e_\mu = A^\nu_\mu Y_\nu, e_a = A^b_a Y_b\}$ and hence any specialized frame can be considered as an adapted one; in particular, any specialized frame is a frame adapted to itself. Obviously (see (6.14)), the coefficients of a connection identically vanish in a given specialized frame considered as an adapted one. This leads to the concept of a *normal frame*, which will be studied on this context in a forthcoming paper. Besides, from the above observation follows that the set of adapted frames coincides with the one of specialized frames.

Exercise 6.2. Verify that the formulae dual to (6.15) and (6.16) are (see (3.4b) and (3.5b))

$$\begin{pmatrix}\tilde{e}^\mu \\ \tilde{e}^a\end{pmatrix} = \begin{pmatrix}[A^\varrho_\tau]^{-1} & 0 \\ -[A^c_d]^{-1}[A^c_\tau][A^\varrho_\tau]^{-1} & [A^c_d]^{-1}\end{pmatrix} \cdot \begin{pmatrix}e^\nu \\ e^b\end{pmatrix} = \begin{pmatrix}([A^\varrho_\tau]^{-1})^\mu_\nu \, e^\nu \\ ([A^c_d]^{-1})^a_b \, e^b - ([A^c_d]^{-1}[A^c_\tau][A^\varrho_\tau]^{-1})^a_\nu \, e^\nu\end{pmatrix} \tag{6.18}$$

$$\begin{pmatrix}\omega^\mu \\ \omega^a\end{pmatrix} \mapsto \begin{pmatrix}\tilde{\omega}^\mu \\ \tilde{\omega}^a\end{pmatrix} = \begin{pmatrix}([A^\varrho_\tau]^{-1}))^\mu_\nu e^\nu \\ ([A^c_d]^{-1})^a_b e^b\end{pmatrix} . \tag{6.19}$$

Example 6.1. If $\{e_I\}$ and $\{\tilde{e}_I\}$ are the frames generated by local coordinates $\{u^I\}$ and $\{\tilde{u}^I\}$, viz. $e_I = \frac{\partial}{\partial u^I}$ and $\tilde{e}_I = \frac{\partial}{\partial \tilde{u}^I}$, the changes (6.16) and (6.17) reduce to (3.31) and (3.32), respectively. The choice $e_I = \frac{\partial}{\partial u^I}$ also reduces definition 6.1 to definition 3.5.

A result similar to proposition 3.3 is valid too provided in its formulation equation (3.32) is replaced with (6.17).

If e_μ has an expansion $e_\mu = e^\nu_\mu \frac{\partial}{\partial u^\nu} + e^b_\mu \frac{\partial}{\partial u^b}$ in the domain U of $\{u^I\} = \{u^\mu = x^\mu \circ \pi, u^a\}$, where $e^b_\mu \colon U \to \mathbb{K}$ and $e^\nu_\mu = x^\nu_\mu \circ \pi$ for some $x^\nu_\mu \colon \pi(U) \to \mathbb{K}$ such that $\det[x^\nu_\mu] \neq 0, \infty$, and we define a frame $\{x_\mu\}$ in $T(\pi(U)) \subseteq T(M)$ by $\{x_\mu := x^\nu_\mu \frac{\partial}{\partial x^\nu}\}$, then

$$\pi_*(X_\mu) = x_\mu, \tag{6.20}$$

by virtue of (3.33) and (3.35). Thus, we have (cf. (3.34))

$$X_\mu = (\pi_*|_{\Delta^h})^{-1}(x_\mu) = (\pi_*|_{\Delta^h})^{-1} \circ \pi_*(e_\mu) \tag{6.21}$$

which can be used in an equivalent definition of a frame $\{X_I\}$ adapted to $\{e_I\}$ (with $\{e_a\}$ being a basis for Δ^v): X_μ should be defined by (6.21) and $X_a = e_a$. If one accepts such a definition of an adapted frame, the 2-index coefficients of a connection should be defined via the equation (6.13a), not by (6.14), and the proofs of some results, like (6.16) and (6.17), should be modified.

Proposition 6.2. *If $\{X_I\}$ is a frame adapted to a frame $\{e_I\}$, with $\{e_a\}$ being a basis for Δ^v, for a C^1 connection Δ^h, then (cf.* (3.36)*)*

$$[X_\mu, X_\nu]_{\_} = R^a_{\mu\nu} X_a + S^\lambda_{\mu\nu} X_\lambda \tag{6.22a}$$

$$[X_\mu, X_b]_{\_} = {}^\circ\Gamma^a_{b\mu} X_a + C^\lambda_{\mu b} X_\lambda \tag{6.22b}$$

$$[X_a, X_b]_{\_} = C^d_{ab} X_d, \tag{6.22c}$$

where (cf. (3.37)*)*

$$\left.\begin{aligned} R^a_{\mu\nu} &:= X_\mu(\Gamma^a_\nu) - X_\nu(\Gamma^a_\mu) - C^a_{\mu\nu} - \Gamma^b_\mu C^a_{\nu b} + \Gamma^b_\nu C^a_{\mu b} \\ &\quad + \Gamma^a_\lambda(-C^\lambda_{\mu\nu} + \Gamma^b_\mu C^\lambda_{\nu b} - \Gamma^b_\nu C^\lambda_{\mu b}) + \Gamma^b_\mu \Gamma^c_\nu C^a_{bd} \\ S^\lambda_{\mu\nu} &:= C^\lambda_{\mu\nu} + \Gamma^b_\mu C^\lambda_{\nu b} - \Gamma^b_\nu C^\lambda_{\mu b} \end{aligned}\right\} \tag{6.23a}$$

$${}^\circ\Gamma^a_{b\mu} := -X_b(\Gamma^a_\mu) - C^a_{\mu b} + \Gamma^d_\mu C^a_{db} - \Gamma^a_\lambda C^\lambda_{\mu b} \tag{6.23b}$$

$$[e_I, e_J]_{\_} =: C^K_{IJ} e_K = C^a_{IJ} e_a + C^\lambda_{IJ} e_\lambda. \tag{6.23c}$$

Proof. Insert (6.13a) into the corresponding commutators, use the definition (6.23c) of the components of the anholonomy object of $\{e_I\}$, and apply (6.13a). Notice, as $\{e_a\}$ is a basis for the integrable distribution Δ^v, we have $[e_a, e_b]_{\_} \in \Delta^v$ and consequently $C^\lambda_{ab} \equiv 0$. □

The functions $R^a_{\mu\nu}$ are the *fibre components of the curvature* of Δ^h and ${}^\circ\Gamma^a_{b\mu}$ are the *fibre coefficients* of Δ^h in the adapted frame $\{X_I\}$; if $e_I = \frac{\partial}{\partial u^I}$ for some bundle coordinates $\{u^I\}$ on E, they reduce to (3.37a) and (3.37b), respectively. From (6.22), we immediately derive

Corollary 6.1. *A connection Δ^h is integrable iff in some (and hence any) adapted frame:*

$$R^a_{\mu\nu} = 0. \tag{6.24}$$

Corollary 6.2. *An adapted frame $\{X_I\}$ is (locally) holonomic iff in it*

$$R^a_{\mu\nu} = {}^\circ\Gamma^a_{b\mu} = S^\lambda_{\mu\nu} = C^d_{ab} = C^\lambda_{\mu b} = 0. \tag{6.25}$$

If the initial frame $\{e_I\}$ is changed into (6.15), then the transformation laws of the quantities (6.23) follow from (6.22) and (6.16); in particular, the curvature components transform according to the tensor equation (3.24b).

Let us now pay attention to the case when (E, π, M) is a *vector* bundle endowed with a connection Δ^h.

According to the above-said in this section, any *adapted* frame $\{X_I\} = \{X_\mu, X_a\}$ in $T(E)$ is equivalent to a pair of frames in $T(M)$ and E according to

$$\{X_\mu, X_a\} \leftrightarrow (\{E_\mu = \pi_*|_{\Delta^h}(X_\mu)\}, \{E_a = v^{-1}(X_a)\}). \tag{6.26}$$

Therefore the vertical and horizontal lifts are given by (cf. lemma 4.1, (4.8a) and (4.11))

$$\operatorname{Sec}(E, \pi, M) \ni Y = Y^a E_a \xrightarrow{v} v(Y) := Y^v = (Y^a \circ \pi) X_a \in \Delta^v \tag{6.27a}$$

$$\mathcal{X}(M) \ni F = F^\mu E_\mu \xrightarrow{h} h(F) := F^h = (F^\mu \circ \pi) X_\mu \in \Delta^h. \tag{6.27b}$$

Thus, we have the linear isomorphism

$$\begin{aligned} (h, v)\colon \mathcal{X}(M) \times \operatorname{Sec}(E, \pi, M) &\to \mathcal{X}(E) \\ (h, v)\colon (F, Y) &\mapsto (F^h, Y^v) \end{aligned} \tag{6.28}$$

which explains why the covariant derivatives (see definition 4.2) represent an equivalent description of the linear connections in vector bundles. Since any vector field $\xi = (\xi^I \circ \pi) X_I \in \mathcal{X}(E)$ has a unique decomposition $\xi = \xi^h \oplus \xi^v$, with $\xi^h = (\xi^\mu \circ \pi) X_\mu$ and $\xi^v = (\xi^a \circ \pi) X_a$, we have

$$(h, v)^{-1}(\xi) = (\pi_*|_{\Delta^h}(\xi^h), v^{-1}(\xi^v)) = (\xi^\mu E_\mu, \xi^a E_a). \tag{6.29}$$

Suppose $\{X_I\}$ and $\{\tilde{X}_I\}$ are two adapted frames. Then they are connected by (cf. (6.3a) and (6.16))

$$\tilde{X}_\mu = (B^\nu_\mu \circ \pi) X_\nu \quad \tilde{X}_a = (B^b_a \circ \pi) X_b, \tag{6.30}$$

where $[B^\nu_\mu]$ and $[B^b_a]$ are some non-degenerate matrix-valued functions on M. The pairs of frames corresponding to them, in accordance with (6.26), are related via

$$\tilde{E}_\mu = B^\nu_\mu E_\nu \quad \tilde{E}_a = B^b_a E_b \tag{6.31}$$

and *vice versa*.

Proposition 6.3. *Let Δ^h be a* linear *connection on a* vector *bundle (E,π,M) and $\{X_\mu\}$ be the frame adapted for Δ^h to a frame $\{e_I\}$ such that $\{e_a\}$ is a basis for Δ^v and*

$$(e_\mu,e_a)|_U = (\partial_\nu,\partial_b)\cdot\begin{bmatrix} B^\nu_\mu\circ\pi & 0 \\ (B^b_{c\mu}\circ\pi)\cdot E^c & B^b_a\circ\pi\end{bmatrix} = \bigl((B^\nu_\mu\circ\pi)\partial_\nu + ((B^b_{c\mu}\circ\pi)\cdot E^c)\partial_b,(B^b_a\circ\pi)\partial_b\bigr), \tag{6.32}$$

where $\partial_I := \frac{\partial}{\partial u^I}$ for some local bundle coordinates $\{u^I\}=\{u^\mu = x^\mu\circ\pi, u^b = E^b\}$ on $U\subseteq E$, $[B^\nu_\mu]$ and $[B^b_a]$ are non-degenerate matrix-valued functions on U, $B^b_{c\mu}\colon U\to\mathbb{K}$, and $\{E^a\}$ is the coframe dual to $\{E_a = v^{-1}(X_a)\}$. Then the 2-index coefficients Γ^a_μ of Δ^h in $\{X_I\}$ have the representation (cf. (4.22)*)*

$$\Gamma^a_\mu = -(\Gamma^a_{b\mu}\circ\pi)\cdot E^b \tag{6.33}$$

on U for some functions $\Gamma^a_{b\mu}\colon U\to\mathbb{K}$, called 3-index coefficients of Δ^h in $\{X_I\}$.

Remark 6.1. The representation (6.33) is not valid for frames more general than the ones given by (6.32). Precisely, equation (6.33) is valid if and only if (6.32) holds for some local coordinates $\{u^I\}$ on U — see (6.17).

Proof. Writing (6.17) for the transformation $\{\partial_I\}\mapsto\{e_I\}$, with $\{e_I\}$ given by (6.32), we get (6.33) with

$$\Gamma^a_{b\mu} = ([B^e_d]^{-1})^a_c({}^\partial\Gamma^c_{b\nu}B^\nu_\mu + B^c_{b\mu}),$$

where ${}^\partial\Gamma^a_{b\nu}$ are the 3-index coefficients of Δ^h in the frame adapted to the coordinates $\{u^I\}$ (see (4.22)). □

Let $\{X_I\}$ and $\{\tilde X_I\}$ be frames adapted to $\{e_I\}$ and $\{\tilde e_I\}$, respectively, with (cf. (6.32))

$$(\tilde e_\mu,\tilde e_a) = (e_\nu,e_b)\cdot\begin{bmatrix} B^\nu_\mu\circ\pi & 0 \\ (B^b_{c\mu}\circ\pi)\cdot E^c & B^b_a\circ\pi\end{bmatrix}, \tag{6.34}$$

in which Δ^h admits 3-index coefficients. Then, due to (6.17) and (6.33), the 3-index coefficients $\Gamma^a_{b\mu}$ and $\tilde\Gamma^a_{b\mu}$ of Δ^h in respectively $\{X_I\}$ and $\{\tilde X_I\}$ are connected by (cf. (4.32))

$$\tilde\Gamma^a_{b\mu} = \bigl([B^e_f]^{-1}\bigr)^a_c(\Gamma^c_{d\nu}B^\nu_\mu + B^c_{d\mu})B^d_b. \tag{6.35}$$

Exercise 6.3. Prove that the transformation $\{e_I\}\mapsto\{\tilde e_I\}$, with $\{\tilde e_I\}$ given by (6.34), is the most general one that preserves the existence of 3-index coefficients of Δ^h provided they exist in $\{e_I\}$.

Introducing the matrices $\Gamma_\mu := [\Gamma^a_{b\mu}]^{n+r}_{a,b=n+1}$, $\tilde\Gamma_\mu := [\tilde\Gamma^a_{b\mu}]^{n+r}_{a,b=n+1}$, $B := [B^a_b]$, and $B_\mu := [B^a_{b\mu}]$, we rewrite (6.35) as (cf. (4.32′))

$$\tilde\Gamma_\mu = B^{-1}\cdot(\Gamma_\nu B^\nu_\mu + B_\mu)\cdot B. \tag{6.35′}$$

A little below (see the text after equation (6.37)), we shall prove that the compatibility of the developed formalism with the theory of covariant derivatives requires further restrictions on the general transformed frames (6.15) to the ones given by (6.34) with

$$B_\mu = \tilde E_\mu(B)\cdot B^{-1} = B^\nu_\mu E_\nu(B)\cdot B^{-1}, \tag{6.36}$$

where $\tilde E_\mu := \pi_*|_{\Delta^h}(\tilde X_\mu) = \pi_*|_{\Delta^h}((B^\nu_\mu\circ\pi)X_\nu) = B^\nu_\mu E_\nu$. In this case, (6.35′) reduces to (cf. (4.32′))

$$\tilde\Gamma_\mu = B^\nu_\mu B^{-1}\cdot(\Gamma_\nu\cdot B + E_\nu(B)) = B^\nu_\mu(B^{-1}\cdot\Gamma_\nu - E_\nu(B^{-1}))\cdot B. \tag{6.37}$$

At last, a few words on the covariant derivative operators ∇ are in order. Without lost of generality, we define such an operator (4.42) via the equations (4.48). Suppose $\{E_\mu\}$ is a

basis for $\mathcal{X}(M)$ and $\{E_a\}$ is a one for $\mathrm{Sec}^1(E,\pi,M)$. Define the *components* $\Gamma^a_{b\mu}\colon M \to \mathbb{K}$ of ∇ in the pair of frames $(\{E_\mu\},\{E_a\})$ by (cf. (4.49))

$$\nabla_{E_\mu}(E_b) = \Gamma^a_{b\mu} E_a. \tag{6.38}$$

Then (4.48) imply (cf. (4.45))

$$\nabla_F Y = F^\mu (E_\mu(Y^a) + \Gamma^a_{b\mu} Y^b) E_a$$

for $F = F^\mu E_\mu \in \mathcal{X}(M)$ and $Y = Y^a E_a \in \mathrm{Sec}^1(E,\pi,M)$. A change $(\{E_\mu\},\{E_a\}) \mapsto (\{\tilde{E}_\mu\},\{\tilde{E}_a\})$, given via (6.31), entails

$$\Gamma^a_{b\mu} \mapsto \tilde{\Gamma}^a_{b\mu} = B^\nu_\mu \big([B^e_f]^{-1}\big)^a_c (\Gamma^c_{d\nu} B^d_b + E_\nu(B^c_b)), \tag{6.39}$$

as a result of (6.38). In a more compact matrix form, the last result reads

$$\tilde{\Gamma}_\mu = B^\nu_\mu B^{-1} \cdot (\Gamma_\nu \cdot B + E_\nu(B)) \tag{6.39$'$}$$

with $\Gamma_\mu := [\Gamma^a_{b\mu}]$, $\tilde{\Gamma}_\mu := [\tilde{\Gamma}^a_{b\mu}]$, and $B := [B^a_b]$.

Thus, if we identify the 3-index coefficients of Δ^h, defined by (6.33), with the components of ∇, defined by (6.38), [25] then the quantities (6.35$'$) and (6.39$'$) must coincide, which immediately leads to the equality (6.36). Therefore

$$(e_\mu, e_a) \mapsto (\tilde{e}_\mu, \tilde{e}_a) = (e_\nu, e_b) \cdot \begin{bmatrix} B^\nu_\mu \circ \pi & 0 \\ \big((B^\nu_\mu E_\nu(B^b_d)(B^{-1})^d_c) \circ \pi\big) E^c & B^b_a \circ \pi \end{bmatrix}\Bigg|_{B=[B^b_a]} \tag{6.40}$$

is the most general transformation between frames in $T(E)$ such that the frames adapted to them are compatible with the linear connection and the covariant derivative corresponding to it. In particular, such are all frames $\{\frac{\partial}{\partial u^I}\}$ in $T(E)$ induced by vector bundle coordinates $\{u^I\}$ on E — see (4.30) and (3.1)–(3.3); the rest members of the class of frames mentioned are obtained from them via (6.40) with $e_I = \frac{\partial}{\partial u^I}$ and non-degenerate matrix-valued functions $[B^\nu_\mu]$ and B.

If $\{X_I\}$ (resp. $\{\tilde{X}_I\}$) is the frame adapted to a frame $\{e_I\}$ (resp. $\{\tilde{e}_I\}$), then the change $\{e_I\} \mapsto \{\tilde{e}_I\}$, given by (6.40), entails $\{X_I\} \mapsto \{\tilde{X}_I\}$ with $\{\tilde{X}_I\}$ given by (6.30) (see (6.15) and (6.16)). Since the last transformation is tantamount to the change

$$(\{E_\mu\},\{E_a\}) \mapsto (\{\tilde{E}_\mu\},\{\tilde{E}_a\}) \tag{6.41}$$

of the basis of $\mathcal{X}(M) \times \mathrm{Sec}(E,\pi,M)$ corresponding to $\{X_I\}$ via the isomorphism (6.28) (see (6.26), (6.30), and (6.31)), we can say that the transition (6.41) induces the change (6.39) of the 3-index coefficients of the connection Δ^h. Exactly the same is the situation one meets in the literature [2, 5, 23] when covariant derivatives are considered (and identified with connections).

Regardless that the change (6.40) of the frames in $T(E)$ looks quite special, it is the most general one that, through (6.16) and (6.26), is equivalent to an arbitrary change (6.41) of a basis in $\mathcal{X}(M) \times \mathrm{Sec}(E,\pi,M)$, i.e. of a pair of frames in $T(M)$ and E.

We would like to remark that, in the general case, equation (4.53) also holds with $F = F^\mu E_\mu$, $G = G^\mu E_\mu$, and

$$(R(E_\mu, E_\nu))(E_b) = R^a_{b\mu\nu} E_a, \tag{6.42}$$

so that

$$R^a_{b\mu\nu} = E_\mu(\Gamma^a_{b\nu}) - E_\nu(\Gamma^a_{b\mu}) - \Gamma^c_{b\mu}\Gamma^a_{c\nu} + \Gamma^c_{b\nu}\Gamma^a_{c\mu} - \Gamma^a_{b\lambda} C^\lambda_{\mu\nu}, \tag{6.43}$$

where the functions $C^\lambda_{\mu\nu}$ define the anholonomy object of $\{E_\mu\}$ via $[E_\mu, E_\nu]_{\_} =: C^\lambda_{\mu\nu} E_\lambda$.

The above results, concerning linear connections on vector bundles, can be generalized for affine connections on vector bundles. For instance, the analogue of propositions 6.3 reads.

[25] Such an identification is justified by the definition of ∇ via the parallel transport assigned to Δ^h (see proposition 4.4) or via a projection, generated by Δ^h, of a suitable Lie derivative on $\mathfrak{X}(E)$ (see definition 4.2).

Proposition 6.4. *Let Δ^h be an affine connection on a vector bundle (E,π,M) and $\{X_\mu\}$ be the frame adapted for Δ^h to a frame $\{e_I\}$ such that $\{e_a\}$ is a basis for Δ^v and*

$$(e_\mu, e_a)|_U = (\partial_\nu, \partial_b)\cdot \begin{bmatrix} B^\nu_\mu \circ \pi & 0 \\ (B^b_{c\mu}\circ\pi)\cdot E^c & B^b_a\circ\pi \end{bmatrix} = \big((B^\nu_\mu\circ\pi)\partial_\nu + ((B^b_{c\mu}\circ\pi)\cdot E^c)\partial_b, (B^b_a\circ\pi)\partial_b\big), \tag{6.44}$$

where $\partial_I := \frac{\partial}{\partial u^I}$ for some local bundle coordinates $\{u^I\} = \{u^\mu = x^\mu\circ\pi, u^b = E^b\}$ on $U\subseteq E$, $[B^\nu_\mu]$ and $[B^b_a]$ are non-degenerate matrix-valued functions on U, $B^b_{c\mu}\colon U\to\mathbb{K}$, and $\{E^a\}$ is the coframe dual to $\{E_a = v^{-1}(X_a)\}$. Then the 2-index coefficients Γ^a_μ of Δ^h in $\{X_I\}$ have the representation (cf. (4.65)*)*

$$\Gamma^a_\mu = -(\Gamma^a_{b\mu}\circ\pi)\cdot E^b + G^a_\mu\circ\pi \tag{6.45}$$

on U for some functions $\Gamma^a_{b\mu}, G^a_\mu\colon U\to\mathbb{K}$.

Remark 6.2. The representation (6.45) is not valid for frames more general than the ones given by (6.44). Precisely, equation (6.45) is valid if and only if (6.44) holds for some local coordinates $\{u^I\}$ on U — see (6.17).

Proof. Writing (6.17) for the transformation $\{\partial_I\}\mapsto\{e_I\}$, with $\{e_I\}$ given by (6.44), we get (6.45) with

$$\Gamma^a_{b\mu} = ([B^e_d]^{-1})^a_c({}^\partial\Gamma^c_{b\nu}B^\nu_\mu + B^c_{b\mu}) \quad G^a_\mu = ([B^e_d]^{-1})^a_b\,{}^\partial G^b_\nu B^\nu_\mu$$

where ${}^\partial\Gamma^a_{b\nu}$ and ${}^\partial G^b_\nu$ are defined via the 2-index coefficients ${}^\partial\Gamma^a_\mu$ of Δ^h in the frame adapted to the coordinates $\{u^I\}$ via ${}^\partial\Gamma^a_\mu = -({}^\partial\Gamma^a_{b\mu}\circ\pi)\cdot E^b + {}^\partial G^a_\mu\circ\pi$ (see theorem 4.3). □

Let $\{X_I\}$ and $\{\tilde X_I\}$ be frames adapted to $\{e_I\}$ and $\{\tilde e_I\}$, respectively, with (cf. (6.44))

$$(\tilde e_\mu, \tilde e_a) = (e_\nu, e_b)\cdot\begin{bmatrix} B^\nu_\mu\circ\pi & 0 \\ (B^b_{c\mu}\circ\pi)\cdot E^c & B^b_a\circ\pi\end{bmatrix}, \tag{6.46}$$

in which (6.45) holds for Δ^h. Then, due to (6.17) and (6.45), the pairs $(\Gamma^a_{b\mu}, G^a_\mu)$ and $(\tilde\Gamma^a_{b\mu}, \tilde G^a_\mu)$ for Δ^h in respectively $\{X_I\}$ and $\{\tilde X_I\}$ are connected by (cf. (4.65) and (4.66))

$$\tilde\Gamma^a_{b\mu} = \big([B^e_f]^{-1}\big)^a_c(\Gamma^c_{d\nu}B^\nu_\mu + B^c_{d\mu})B^d_b \tag{6.47a}$$

$$\tilde G^a_\mu = \big([B^e_f]^{-1}\big)^a_b G^b_\nu B^\nu_\mu \tag{6.47b}$$

Exercise 6.4. Prove that the transformation $\{e_I\}\mapsto\{\tilde e_I\}$, with $\{\tilde e_I\}$ given by (6.46), is the most general one that preserves the existence of the relation (6.45) for Δ^h provided it holds in $\{e_I\}$.

Further one can repeat *mutatis mutandis* the text after exercise 6.3 to the paragraph containing equation (6.41) including.

7. Conclusion

In this paper we have presented a short (and partial) review of (one of the approaches to) the connection theory on bundles whose base and bundle spaces are (C^2) differentiable manifolds. Special attention was paid to connections, in particular linear ones, on vector bundles, which find wide applications in physics [21, 32]. However, many other approaches, generalizations, alternative descriptions, particular methods, etc. were not mentioned at all. In particular, these include: connections on more general (e.g. topological) bundles, connections on principal bundles (which are important in the gauge field theories), holonomy groups, flat

connections, Riemannian connections, etc., etc. The surveys [33, 34] contain essential information on these and many other items. Consistent and self-contained exposition of such problems can be found in [22, 23, 35, 36].

If additional geometric structures are added to the theory considered in Sect. 3, there will become important connections compatible with these structures. In this way arise many theories of particular connections; we have demonstrated that on the example of linear connections on vector bundles (Sect. 4). Here are two more such cases.

If a free right action $R\colon g \mapsto R_g\colon E \to E$, $g \in G$, of a Lie group G on the bundle space E of a bundle (E, π, M) is given and $\pi\colon E \to M = E/G$ is the canonical projection, we have a principal bundle (E, π, M, G). The connections that respect the right action R are the most important ones on principal bundles. Such a connection Δ^h is defined by definition 3.1 to which the condition

$$(R_g)_*(\Delta^h_p) = \Delta^h_{R_g(p)} \qquad g \in G \quad p \in E \tag{7.1}$$

is added and is called a principal connection. Alternatively, one can require the parallel transport P generated by Δ^h to commute with R, viz.

$$R_g \circ \mathsf{P}^\gamma = \mathsf{P}^\gamma \circ R_g \qquad g \in G \quad \gamma\colon [\sigma, \tau] \to M. \tag{7.2}$$

The theory of connections satisfying (7.1) is very well developed; see, e.g., [2, 36].

Suppose a real bundle (E, π, M) is endowed with a bundle metric g, i.e. $g\colon x \mapsto g_x$, $x \in M$, with $g_x\colon \pi^{-1}(x) \times \pi^{-1}(x) \to \mathbb{R}$ being bilinear and non-degenerate mapping for all $x \in M$. The equality

$$g_{\gamma(\sigma)} = g_{\gamma(\tau)} \circ \big(\mathsf{P}^\gamma \times \mathsf{P}^\gamma\big) \qquad \gamma\colon [\sigma, \tau,] \to M, \tag{7.3}$$

which expresses the preservation of the g-scalar products by the parallel transport P assigned to a connection Δ^h, specifies the class of g-compatible (metric-compatible) connections on (E, π, M). Such are the Riemannian connections on a Riemannian manifold M, which are g-compatible connections on the tangent bundle $(T(M), \pi_T, M)$; see, for instance, [2, 23].

The consideration of arbitrary (co)frames in Sect. 6 may seem slightly artificial as the general theory can be developed without them. However, this is not the generic case when one starts to apply the connection theory for investigation of particular problems. It may happen that some problem has solutions in general (co)frames but it does not possess solutions when (co)frames generated directly by local coordinates are involved. For example [37], local coordinates (holonomic frames) normal at a given point for a covariant derivative operator (linear connection) ∇ on a manifold exist if ∇ is torsionless at that point, but anholonomic frames normal at a given point for ∇ exist always.

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